ASYMPTOTICALLY HYPERBOLIC MANIFOLDS WITH SMALL MASS

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ABSTRACT. For asymptotically hyperbolic manifolds of dimension n with scalar curvature at least equal to $-n(n-1)$ the conjectured positive mass theorem states that the mass is non-negative, and vanishes only if the manifold is isometric to hyperbolic space. In this paper we study asymptotically hyperbolic manifolds which are also conformally hyperbolic outside a ball of fixed radius, and for which the positive mass theorem holds. For such manifolds we show that the conformal factor tends to one as the mass tends to zero.

CONTENTS

1. INTRODUCTION

The mass of an asymptotically hyperbolic Riemannian manifold is a geometric invariant which has been introduced by Wang [26] and Chruściel and Herzlich [11] using slightly different approaches. The mass is computed in a fixed asymptotically hyperbolic end and gives a measure of the leading order deviation of the geometry

from a hyperbolic background metric in the end. For the family of Anti-de Sitter-Schwarzschild metrics the mass coincides with the mass parameter.

In both papers mentioned above, a positive mass theorem is proved for spin manifolds using an adaptation of Witten's spinor argument [27]. This theorem states that a complete asymptotically hyperbolic spin manifold of dimension n must have non-negative mass if its scalar curvature is at least equal to $-n(n-1)$ (which is the scalar curvature of hyperbolic space). Previous work in the physics literature include [1], [13], [4].

The positive mass theorem also contains a rigidity statement saying that the mass vanishes if and only if the manifold is isometric to hyperbolic space. In Witten's spinor argument the rigidity follows from the fact that vanishing mass forces a certain spinor field to satisfy the overdetermined Killing equation, which implies that the manifold is hyperbolic. Without the spin assumption the positive mass theorem for asymptotically hyperbolic manifolds is still open. Partial results have been obtained by Anderson, Cai, and Galloway in [2] where an adaptation of the minimal surface method of [23] is used, see also [17, Section $5\frac{5}{6}$]. In [2] the rigidity in the case of vanishing mass is proved by first showing that the manifold is Einstein. This is done by an argument involving a deformation of the metric by the traceless Ricci tensor, if this is non-zero one can deform to a metric with strictly negative mass which gives a contradiction.

With the rigidity statement of the positive mass theorem in mind, it is natural to ask what happens if the mass is close to zero and scalar curvature at least equal to $-n(n-1)$. Must the manifold then be close to hyperbolic space in some appropriate sense? Such a statement can never hold true globally, as can be seen by the example of the Anti-de Sitter-Schwarzschild metric.

The same question can be asked in relation to the rigidity part of the positive mass theorem for asymptotically Euclidean manifolds: must an asymptotically Euclidean with small mass and non-negative scalar curvature be close to Euclidean space in some sense? Asymptotically Euclidean spin manifolds with small mass have been studied by Bray and Finster, see [7] and [12]. From estimates on the spinor field in Witten's argument they find that the L^2-norm of the curvature tensor (over the manifold minus an exceptional set) is bounded in terms of the mass. Lee [21] studies asymptotically Euclidean manifolds which are conformally flat outside a compact set. For such manifold he proves that the conformal factor can be controlled by the mass, so that the conformal factor tends to one as the mass tends to zero. The argument by Lee does not need the manifold to be spin, but it needs the assumption that the positive mass theorem holds for any asymptotically Euclidean metric on the manifold.

In the present paper we will adapt the ideas of Lee to the setting of asymptotically hyperbolic manifolds. We define a class of n-dimensional asymptotically hyperbolic manifolds (M, g) which have scalar curvature greater than $-n(n-1)$ and have a chart at infinity $\Phi : M \setminus K \to \mathbb{H}^n \setminus B_{R_0}$, where K is a compact subset of M and R_0 is a given fixed radius. Also $\Phi_* g$ is conformal to the hyperbolic metric,

$$\Phi_* g = U^\kappa b,$$

where $\kappa = \frac{4}{n-2}$, and $\mathrm{Scal}^g = -n(n-1)$ on $M \setminus K$. We must assume further that the positive mass theorem holds for any asymptotically hyperbolic metric on the manifold M. We prove that given any $\varepsilon > 0$, there is a $\delta > 0$ such that if a metric belongs to this class and has mass $m < \delta$ then the conformal factor U satisfies

$|U - 1| < \varepsilon$. We refer the reader to Definition 3.1 and Theorem A for precise statements of the results.

The overall strategy follows [21], we define a 1-parameter family of asymptotically hyperbolic metrics involving a geometric property of (M, g) and look at their mass. If the mass of (M, g) is close to zero and if it varies too widely with respect to the parameter, this yields a contradiction with the positive mass theorem. The geometric property of (M, g) we use measures how far the metric g is from being static (see Proposition 3.11). This is a refinement of the deformation argument for rigidity in [2], it allows us in a sense to localize the deformation of the metric in a compact subset.

We also give a simpler proof for spin manifolds, see Theorem B. This argument is based on the fact that the mass controls a certain functional which measures how close (M, g) is to allow a Killing spinor, and this functional in turn depends continuously on the conformal factor U.

The most stringent assumption of our theorem is probably that the metric must be conformal to the hyperbolic metric outside a compact subset. However, a result of Chruściel and Delay [10, Proposition 6.2] tells us that every asymptotically hyperbolic manifold of constant scalar curvature can be approximated by metrics which are conformal to the hyperbolic metric outside a ball.

The small mass theorem of Lee [21] appears as an ingredient in the proof of the Penrose inequality by Bray [6] and Bray and Lee [8]. In a forthcoming work we plan to adress the adaptation of Bray's proof of the Penrose inequality to the case of asymptotically hyperbolic manifolds.

This paper is organized as follows. In Section 2 we give the definitions of asymptotically hyperbolic manifolds and their mass. Section 3 begins with some results on the conformal factors at infinity for manifolds in $\mathcal{A}(R_0)$, thereafter we show that we can reduce to the case $\mathrm{Scal} = -n(n - 1)$ everywhere by a conformal change while controlling the mass. We then give the argument of our main theorem, with proofs of the more technical lemmas in a separate subsection. In Section 4 we give the alternative argument for spin manifolds. Finally, in Appendix A we give details of the Anti-de Sitter-Schwarzschild metric which are used in the paper.

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2. Preliminaries

2.1. **The mass of an asymptotically hyperbolic manifold.** Following the work of Chruściel and Herzlich, [11] and [18], we define the mass of an asymptotically hyperbolic manifold. For conformally compact manifolds the definition of the asymptotically hyperbolic mass coincides with the mass introduced by Wang in [26]. In this paper we denote n-dimensional hyperbolic space by $\mathbb{H}^n$ and its metric is denoted by b. We fix a point in $\mathbb{H}^n$ as origin, in polar coordinates around this point we have $b = dr^2 + \sinh^2 r\sigma$ on $(0, \infty) \times S^{n-1}$ where σ denotes the standard round metric on S^{n-1}. The open ball of radius R centered at the origin is denoted by B_R and its closure is denoted by $\overline{B}_R$.

Let $\mathcal{N} := \{V \in C^\infty(\mathbb{H}^n) \mid \mathrm{Hess}^b V = Vb\}$. This is a vector space with a basis consisting of the functions

$$V_{(0)} = \cosh r, \; V_{(1)} = x^1 \sinh r, \ldots, V_{(n)} = x^n \sinh r,$$

where the functions $x^1, \ldots, x^n$ are the coordinate functions on $\mathbb{R}^n$ restricted to S^{n-1}. The vector space $\mathcal{N}$ is equipped with an inner product η of Lorentzian signature characterized by the condition that the basis above is orthonormal, $\eta(V_{(0)}, V_{(0)}) = 1$, and $\eta(V_{(i)}, V_{(i)}) = -1$ for $i = 1, \ldots, n$. We give $\mathcal{N}$ a time orientation by specifying the vector $V_{(0)}$ to be future directed. The subset $\mathcal{N}^+$ of positive functions then coincides with the interior of the future lightcone. Also we denote by $\mathcal{N}^1$ the subset of $\mathcal{N}^+$ of functions V with $\eta(V, V) = 1$, this is the unit hyperboloid in the future lightcone of $\mathcal{N}$. For a point $p_0 \in \mathbb{H}^n$ the function

$$V := \cosh d_b(p, \cdot)$$

is in $\mathcal{N}^1$, and any function in $\mathcal{N}^1$ can be given in this form.

A Riemannian manifold (M, g) is called *asymptotically hyperbolic* if there is a compact subset $K \subset M$ and a diffeomorphism $\Phi : M \setminus K \to \mathbb{H}^n \setminus \overline{B}_R$ for which $\Phi_* g$ and b are uniformly equivalent on $\mathbb{H}^n \setminus B_R$ and

$$\int_{\mathbb{H}^n \setminus B_R} \left(|e|^2 + |\nabla^b e|^2 \right) \cosh r \, d\mu^b < \infty, \tag{1a}$$

$$\int_{\mathbb{H}^n \setminus B_R} |s^g + n(n-1)| \cosh r \, d\mu^b < \infty, \tag{1b}$$

where $e := \Phi_* g - b$ and r is the (hyperbolic) distance from an arbitrary given point in $\mathbb{H}^n$. The diffeomorphism Φ is also called a chart, or a set of coordinates, at infinity.

The linear functional H_Φ on $\mathcal{N}$ defined by

$$H_\Phi(V) = H_\Phi^g(V) = \lim_{r \to \infty} \int_{S_r} \left(V(\operatorname{div}^b e - d \operatorname{tr}^b e) + (\operatorname{tr}^b e) dV - e(\nabla^b V, \cdot) \right) (\nu_r) \, d\mu^b$$

is called the *mass functional* of (M, g) with respect to Φ. Proposition 2.2 of [11] tells us that the limit involved in the definition of H_Φ exists and is finite when the decay conditions (1a)-(1b) are satisfied. If Φ is a chart at infinity as above and A is an isometry of the hyperbolic metric b then $A \circ \Phi$ is again a chart at infinity and it is not complicated to check that

$$H_{A \circ \Phi}(V) = H_\Phi(V \circ A^{-1}).$$

If Φ_1, Φ_2 are charts at infinity as above, then [18, Theorem 2.3] tells us that there is an isometry A of b so that $\Phi_2 = A \circ \Phi_1$ modulo lower order terms which do not affect the mass functional.

The mass functional H_Φ is timelike future directed if $H_\Phi(V) > 0$ for all $V \in \mathcal{N}^+$. In this case the *mass* of the asymptotically hyperbolic manifold (M, g) is defined by

$$m^g := \frac{1}{2(n-1)\omega_{n-1}} \inf_{\mathcal{N}^1} H_\Phi^g(V).$$

Here ω_{n-1} denotes the volume of the sphere (S^{n-1}, σ). The factor in front of the infimum is such that the mass of the space-like slice

$$g_{\text{AdSS}} = \frac{d\rho^2}{1 + \rho^2 - \frac{2m}{\rho^{n-2}}} + \rho^2 \sigma$$

of the Anti-de Sitter-Schwarzschild metric is equal to the parameter m in the metric. Note that Chruściel and Herzlich [11, (3.5) and (3.6)] define m^g without this factor. If H_Φ^g is timelike future directed we may replace the coordinates at infinity Φ by $A \circ \Phi$

for a suitably chosen isometry A so that $m^g = \frac{1}{2(n-1)\omega_{n-1}} H^g_\Phi(V_{(0)})$. Coordinates with this property are called *balanced*.

The positive mass theorem for asymptotically hyperbolic manifolds, [11, Theorem 4.1] and [26, Theorem 1.1], states that the mass functional is timelike future directed or zero for complete asymptotically hyperbolic spin manifolds with scalar curvature $\mathrm{Scal} \geq -n(n-1)$. In [2, Theorem 1.3] the same result is proved with the spin assumption replaced by assumptions on the dimension and on the geometry at infinity.

2.2. Conformally hyperbolic metrics.
We now compute the mass functional of a metric g which is asymptotically hyperbolic and conformal to the hyperbolic metric in the chart at infinity. That is $g = U^\kappa \Phi^* b$ where U is a positive function and we set $\kappa := \frac{4}{n-2}$ as we do throughout the paper. In this case $e = fb$ where $f := (U^\kappa - 1)$. The metric g is asymptotically hyperbolic if e satisifes (1a)-(1b), which turns into weighted integral conditions on U and its first two derivatives. The mass functional becomes

$$H^g_\Phi(V) = (n-1) \lim_{r \to \infty} \int_{S_r} (f \partial_r V - V \partial_r f) \, d\mu^b.$$

Assume that the function U has the expansion

$$U = 1 + \frac{2^n}{n+1} v e^{-nr} + O(e^{-(n+1)r})$$

in polar coordinates, where v is a function on S^{n-1}. Then

$$H^g_\Phi \left(\sum_{i=0}^n a_i V_{(i)} \right) = \frac{4(n-1)}{n-2} \int_{S^{n-1}} \left(a_0 + \sum_{i=1}^n a_i x^i \right) v \, d\mu^\sigma,$$

and in particular we have

$$m^g \leq \frac{1}{2(n-1)\omega_{n-1}} H^g_\Phi(V_{(0)}) = \frac{2}{(n-2)\omega_{n-1}} \int_{S^{n-1}} v \, d\mu^\sigma \qquad (2)$$

where equality holds if Φ is a balanced chart at infinity.

3. Near equality case of the positive mass theorem: the general case

In this section, we prove an analog of the main result of [21]. We first introduce the following class of asymptotically hyperbolic manifolds.

Definition 3.1. For $R_0 > 0$ we let $\mathcal{A}(R_0)$ be the class of 4-tuples (M, g, Φ, U) such that

- (M, g) is a complete asymptotically hyperbolic Riemannian manifold with respect to Φ, where Φ is a diffeomorphism from the exterior of a compact set $K \subset M$ to $\mathbb{H}^n \setminus \overline{B}_{R_0}$,
- $\mathrm{Scal}^g \geq -n(n-1)$, and $\mathrm{Scal}^g = -n(n-1)$ on $M \setminus K$,
- U is a positive function on $\mathbb{H}^n \setminus B_{R_0}$ such that $U \to 1$ infinity and $\Phi_* g = U^\kappa b$,
- the coordinates at infinity Φ are balanced,
- the positive mass theorem holds for any asymptotically hyperbolic metric on M.

We will prove the following theorem concerning the near-equality case for the positive mass theorem.

Theorem A. *Let $R_1 > R_0$ and $\varepsilon > 0$. Then there is a constant $\delta > 0$ so that*

$$|U - 1| \leq \varepsilon e^{-nr}$$

on $\mathbb{H}^n \setminus B_{R_1}$ for all $(M, g, \Phi, U) \in \mathcal{A}(R_0)$ with $m^g < \delta$.

We fix once and for all the value of R_0 and abbreviate $\mathcal{A} = \mathcal{A}(R_0)$.

3.1. **A priori estimates.** We first prove estimates on the conformal factor U which are valid for any element of $\mathcal{A}$.

Lemma 3.2. *There are positive constants A, A_k, $k = 0, 1, \ldots$, such that for any (M, g, Φ, U) belonging to the class $\mathcal{A}$ we have*

$$\frac{1}{A} \leq U \leq A,$$

$$\left| \nabla^{(k)}(U - 1) \right| \leq A_k e^{-nr} \quad \text{for } k \geq 0,$$

on $\mathbb{H}^n \setminus B_{R_1}$.

Note that these estimates are specific to the case of asymptotically hyperbolic geometry. In the Euclidean context they cannot be true due to the fact that the Yamabe equation (which is then the Laplace equation) is linear.

Proof. The assumption on the scalar curvature of $\Phi_* g = U^\kappa b$ on $\mathbb{H}^n \setminus B_{R_0}$ implies that U solves the Yamabe equation

$$-\frac{4(n-1)}{n-2} \Delta^b U - n(n-1)U = -n(n-1)U^{\kappa+1} \tag{3}$$

on $\mathbb{H}^n \setminus B_{R_0}$. From Propositions A.1 and A.2 we know that there exists a solution U_+ of Equation (3) on $\mathbb{H}^n \setminus \overline{B}_{R_0}$ such that $U_+ = 1 + O(e^{-nr})$ at infinity and $U_+ \to \infty$ on ∂B_{R_0}. Now the same argument as in [14, Proposition 3.6] can be used to show that $U \leq U_+$. Namely, the substitution $U = e^\varphi$ brings equation (3) into the form

$$-\frac{4(n-1)}{n-2}\left(\Delta^b \varphi + |d\varphi|_b^2\right) - n(n-1) = -n(n-1)e^{\kappa\varphi}.$$

Subtracting the respective equations for φ_+ and φ gives

$$-\frac{4(n-1)}{n-2}\left(\Delta^b(\varphi_+ - \varphi) + \langle d(\varphi_+ - \varphi), d(\varphi_+ + \varphi)\rangle_b\right) + n(n-1)\left(e^{\kappa\varphi_+} - e^{\kappa\varphi}\right) = 0,$$

and from the standard maximum principle we conclude that $\varphi_+ \geq \varphi$, hence $U_+ \geq U$. Similarly, from Proposition A.3, there exists a function U_- such that U_- solves Equation (3), $U_- = 1 + O(e^{-nr})$ at infinity, and $U_- = 0$ on ∂B_{R_0}. From the maximum principle we also conclude that $U_- \leq U$.

We can now finish the proof of the lemma. The existence of the constants A and A_0 follows from the fact that $U_- \leq U \leq U_+$ on $\mathbb{H}^n \setminus B_{R_1}$. Finally, since $u = U - 1$ satisfies

$$-\frac{4(n-1)}{n-2}\Delta^b u = -n(n-1)\left((1 + u)^{\kappa+1} - 1\right) + n(n-1)u$$

we can apply elliptic regularity in balls of fixed radius as above and combine with standard bootstrap arguments to get the existence of constants A_k for $k \geq 1$. □

From the estimates in Lemma 3.2 together with (2) we conclude that the mass is uniformly bounded in $\mathcal{A}$.

Corollary 3.3. *There exists a constant $C = C(R_0)$ such that for all elements (M, g, Φ, U) belonging to the class $\mathcal{A}(R_0)$, the mass satisfies $m^g \leq C$.*

Proposition 3.4. *Let $R_1 > R_0$ be a fixed radius. Then there exists a constant $C > 0$ such that for any (M, g, Φ, U) in the class $\mathcal{A}$ we have*

$$|U - 1| \leq C \left(\sup_{\mathbb{H}^n \setminus B_{R_1}} |U - 1| \right) e^{-nr} \tag{4}$$

on $\mathbb{H}^n \setminus B_{R_1}$.

Proof. From Appendix A we know that there exist functions f_{m_+} and f_{m_-} which solve (3) on $\mathbb{H}^n \setminus B_{R_0}$, and satisfy $f_{m_+} \to \infty$ on ∂B_{R_0}, $f_{m_-} = 0$ on ∂B_{R_0}. From the proof of Lemma 3.2 we know that $f_{m_-} \leq U \leq f_{m_+}$ on $\mathbb{H}^n \setminus B_{R_0}$.

Let $0 \leq m \leq m_+$, then f_m with $1 \leq f_m \leq f_{m_+}$ is defined for $r \geq R_1$. From the proof of Proposition A.1 we know that $0 \leq f_m - 1 \leq Cme^{-nr}$ for $r \geq r_1(m) :=$ $\max\{R_1, r((2m)^{1/n})\}$. It is not complicated to extend this estimate to the whole interval $r \geq R_1$. Indeed, let $\mu > 0$ be such that $R_1 = r((2\mu)^{1/n})$. If $0 \leq m \leq \mu$ then we have $r_1(\mu) = R_1$, hence the estimate already holds for $r \geq R_1$. Therefore it suffices to consider the case $\mu < m \leq m_+$ which corresponds to the situation $r_1(m) > R_1$. Since f_m is decreasing we have $f_m - 1 \leq f_m(R_1) - 1 \leq f_{m_+}(R_1) - 1$ on $R_1 \leq r \leq r_1(m)$, whereas $me^{-nr} \geq \mu e^{-nr_1(m)} \geq \mu e^{-nr_1(m_+)}$ on this interval. It is now clear that up to increasing C if necessary, we can assume that the inequality $0 \leq f_m - 1 \leq Cme^{-nr}$ holds for $r \geq R_1$. In the rest of the proof, a constant $C > 0$ might vary from line to line but remains independent of m.

Using Proposition A.3 we can similarly prove that the inequality $Cme^{-nr} \leq f_m - 1 \leq 0$ holds for $r \geq R_1$ in the case when $m_- \leq m \leq 0$. This yields

$$|f_m - 1| \leq C|m|e^{-nr}$$

for $m_- \leq m \leq m_+$ and $r \geq R_1$. Let us now chose $\underline{m}, \overline{m} \in (m_-, m_+)$ so that $f_{\underline{m}}(R_1) = \inf_{\partial B_{R_1}} U$ and $f_{\overline{m}}(R_1) = \sup_{\partial B_{R_1}} U$. Again, the use of the maximum principle as in the proof of Proposition A.1 yields $f_{\underline{m}} \leq U \leq f_{\overline{m}}$ on $\mathbb{H}^n \setminus B_{R_1}$. Consequently, we have the estimate

$$|U - 1| \leq C \max\left\{|\underline{m}|, |\overline{m}|\right\} e^{-nr}$$

on $\mathbb{H}^n \setminus B_{R_1}$.

With all these preliminaries at hand, (4) is a simple consequence of the fact that there exists a constant $C > 0$ such that

$$|m| \leq C|f_m(R_1) - 1| \quad \text{for} \quad m_- \leq m \leq m_+. \tag{5}$$

Indeed, if we assume that this estimate holds, then

$$|U - 1| \leq C \max \left\{ \left| f_{\underline{m}}(R_1) - 1 \right|, |f_{\overline{m}}(R_1) - 1| \right\} e^{-nr}$$

$$= C \max \left\{ \left| \inf_{\partial B_{R_1}} U - 1 \right|, \left| \sup_{\partial B_{R_1}} U - 1 \right| \right\} e^{-nr}$$

$$= C \max \left\{ \left| \inf_{\partial B_{R_1}} (U - 1) \right|, \left| \sup_{\partial B_{R_1}} (U - 1) \right| \right\} e^{-nr}$$

$$\leq C \left(\sup_{\partial B_{R_1}} |U - 1| \right) e^{-nr}$$

$$\leq C \left(\sup_{\mathbb{H}^n \setminus \partial B_{R_1}} |U - 1| \right) e^{-nr}.$$

Consequently, in order to complete the proof, we only need to prove (5). In fact, (5) will follow from the monotonicity of f_m if we show that

$$|m| \leq C|f_m(R_2) - 1| \quad \text{for} \quad m_- \leq m \leq m_+, \tag{6}$$

for some $R_2 > R_1$. We fix $R_2 > \max\{r_0(m_+), R_1\}$ and set $x := f_m(R_2)$. It is clear that $f_{m_-}(R_2) \leq x \leq f_{m_+}(R_2)$ for $m_- \leq m \leq m_+$, and that $r^{-1}(R_2) = x^{\frac{2}{n-2}} \sinh R_2 > a(m)$. Then (26) yields

$$\int_{x^{\frac{2}{n-2}} \sinh R_2}^{\infty} \frac{d\rho}{\rho \sqrt{1 + \rho^2 - \frac{2m}{\rho^{n-2}}}} = \int_{R_2}^{\infty} \frac{dr}{\sinh(r)}.$$

We define

$$F(x, m) := \int_{x^{\frac{2}{n-2}} \sinh R_2}^{\infty} \frac{d\rho}{\rho \sqrt{1 + \rho^2 - \frac{2m}{\rho^{n-2}}}},$$

where $f_{m_-}(R_2) \leq x \leq f_{m_+}(R_2)$, $m_- \leq m \leq m_+$. It is straightforward to check that

$$\frac{\partial F}{\partial m} = \int_{x^{\frac{2}{n-2}} \sinh R_2}^{\infty} \frac{d\rho}{2\rho^{n-1} \left(1 + \rho^2 - \frac{2m}{\rho^{n-2}}\right)^{3/2}}$$

$$\geq \int_{f_{m_+}(R_2)^{\frac{2}{n-2}} \sinh R_2}^{\infty} \frac{d\rho}{2\rho^{n-1} \left(1 + \rho^2 - \frac{2m_-}{\rho^{n-2}}\right)^{3/2}}.$$

is positive and uniformly bounded away from zero, and that

$$\frac{\partial F}{\partial x} = -\frac{2}{(n-2)x \sqrt{1 + x^{\frac{4}{n-2}} (\sinh R_2)^2 - \frac{2m}{x^2 (\sinh R_2)^{n-2}}}}$$

is uniformly bounded. We conclude that there exists $C > 0$ such that $|m'(x)| < C$ for $x \in (f_{m_-}(R_2), f_{m_+}(R_2))$. Finally, applying the mean value theorem we arrive at (6) and thus (5) follows. $\square$

Corollary 3.5. *There exists a radius $R_2 > R_1$ such that for $(M, g, \Phi, U) \in \mathcal{A}$ the function $|U - 1|$ reaches its maximum over $\mathbb{H}^n \setminus B_{R_1}$ in the annulus $A_{R_1, R_2} = \overline{B}_{R_2} \setminus B_{R_1}$.*

Proof. Choose R_2 such that $Ce^{-nR_2} \leq 1$. Then for any point such that $r > R_2$ we have

$$|U - 1| \leq C \left(\sup_{\mathbb{H}^n \setminus B_{R_1}} |U - 1| \right) e^{-nr} < \sup_{\mathbb{H}^n \setminus B_{R_1}} |U - 1|.$$

$\square$

3.2. **Reduction to constant scalar curvature.** The aim of this section is to reduce the proof of Theorem A to the particular case of metrics with constant scalar curvature $\mathrm{Scal}^g = -n(n-1)$. For this we show that the conformal factor transforming the metric g to a metric with constant scalar curvature can be uniformly controlled on $\mathbb{H}^n \setminus B_{R_1}$ by the difference between the masses (more exactly of the time components $H_\Phi(V_{(0)})$ of the mass functional) of the two metrics. This is the content of Proposition 3.6. This reduction turns out to be convenient to obtain estimates in the second part of the proof.

We introduce the restricted class $\mathcal{A}_0(R_0)$ of 4-tuples $(M, g, \Phi, U) \in \mathcal{A}$ such that $\mathrm{Scal}^g = -n(n-1)$ on all of M. In the proof of Theorem A we first show the result for elements of $\mathcal{A}_0 = \mathcal{A}_0(R_0)$, the following lemma will allow us to extend the proof to the larger class $\mathcal{A}$. A similar result can be found in [2, Proposition 3.13].

Proposition 3.6. *Given $(M, g, \Phi, U) \in \mathcal{A}$, there exists a unique positive function w on M such that $\widetilde{g} := w^\kappa g$ is asymptotically hyperbolic with constant scalar curvature $\mathrm{Scal}^{\widetilde{g}} = -n(n-1)$. The metric $\widetilde{g}$ has mass $m^{\widetilde{g}} \leq m^g$. Further, for $p > n/2$ there is a constant $C > 0$ independent of (M, g, Φ, U) such that*

$$\sup_{\mathbb{H}^n \setminus B_{R_1}} \left| U - \widetilde{U} \right| \leq C \left(m^g - m^{\widetilde{g}} \right)^{1/p},$$

where $\widetilde{U} := Uw$.

To prove this lemma we first make a certain observation about $(M, g, \Phi, U) \in \mathcal{A}$. If we set $U = 1 + u$ then (3) can be written in the form

$$u''_{rr} + (n-1) \coth r \, u'_r - nu = f(u) - \sinh^{-2} r \Delta^\sigma u \qquad (7)$$

where

$$f(u) := \frac{n(n-2)}{4} \left((1+u)^{\frac{n+2}{n-2}} - 1 - \frac{n+2}{n-2} u \right).$$

We remark that the ordinary differential equation

$$u''(r) + (n-1) \coth r \, u'(r) - nu(r) = 0$$

has the solutions

$$u_0 = \cosh r \int_r^\infty \frac{1}{\cosh^2 \tau \sinh^{n-1} \tau} \, d\tau = \frac{2^n}{n+1} e^{-nr} + O(e^{-(n+2)r}), \quad u_1 = \cosh r.$$

Lemma 3.7. *Suppose $U = 1 + u$ is such that u satisfies (7) on $\mathbb{H}^n \setminus B_{R_0}$ and the metric $U^\kappa b$ is asymptotically hyperbolic with respect to the identity chart at infinity. Then $v := u/u_0$ satisfies*

$$\int_{S^{n-1}} v(s) \, d\mu^\sigma \geq \frac{(n-2)\omega_{n-1}}{2} m^{U^\kappa b}$$
$$+ \int_s^\infty \left(1 - \frac{\cosh s}{\cosh r} \frac{u_0(r)}{u_0(s)} \right) \cosh r \sinh^{n-1} r \left(\int_{S^{n-1}} f(u(r, \theta)) \, d\mu^\sigma \right) dr$$

where $m^{U^\kappa b}$ is the mass of the metric $U^\kappa b$. Equality holds if the identity chart at infinity is balanced for $U^\kappa b$.

Proof. We observe that $v'_r = O(1)$ by Lemma 3.2. Substituting $u = u_0 v$ into (7) we get

$$u_0 v''_{rr} + (2u'_0 + (n-1)\coth r\, u_0)v'_r = f(u) - \sinh^{-2} r\Delta^\sigma u.$$

If we multiply this equation by $u_0 \sinh^{n-1} r$ we obtain

$$\left(u_0^2 \sinh^{n-1} r v'_r\right)'_r = u_0 \sinh^{n-1} r \left(f(u) - \sinh^{-2} r\Delta^\sigma u\right).$$

Integration from t to ∞ gives

$$\left(u_0^2 \sinh^{n-1} r\, v'_r\right)_{|r=\infty} - \left(u_0^2 \sinh^{n-1} r\, v'_r\right)_{|r=t}$$

$$= \int_t^\infty u_0(r) \sinh^{n-1} r \left(f(u(r,\theta)) - \sinh^{-2} r\Delta^\sigma u(r,\theta)\right)\, dr.$$

Since $u_0^2 \sinh^{n-1} r\, v'_r = O(e^{-(n+1)r})$ the first term in the left hand side vanishes, and consequently we have

$$-v'_r(t,\theta) = \frac{1}{u_0^2(t)\sinh^{n-1} t} \int_t^\infty u_0(r) \sinh^{n-1} r \left(f(u(r,\theta)) - \sinh^{-2} r\Delta^\sigma u(r,\theta)\right)\, dr.$$

Integrating from s to ∞ and changing order of integration we obtain

$$v(s) - \lim_{r\to\infty} v(r)$$

$$= \int_s^\infty \frac{1}{u_0^2(t)\sinh^{n-1} t} \int_t^\infty u_0(r) \sinh^{n-1} r \left(f(u(r,\theta)) - \sinh^{-2} r\Delta^\sigma u(r,\theta)\right)\, dr dt$$

$$= \int_s^\infty \left(\int_s^r \frac{1}{u_0^2(t)\sinh^{n-1} t}\, dt\right) u_0(r) \sinh^{n-1} r \left(f(u(r,\theta)) - \sinh^{-2} r\Delta^\sigma u(r,\theta)\right)\, dr.$$

Here the integral over t is

$$\int_s^r \frac{1}{u_0^2(t)\sinh^{n-1} t}\, dt$$

$$= \int_s^r \frac{1}{\cosh^2 t \sinh^{n-1} t \left(\int_t^\infty \frac{1}{\cosh^2 \tau \sinh^{n-1} \tau}\, d\tau\right)^2}\, dt$$

$$= \int_s^r \left(\frac{1}{\int_t^\infty \frac{1}{\cosh^2 \tau \sinh^{n-1} \tau}\, d\tau}\right)'\, dt$$

$$= \frac{1}{\int_r^\infty \frac{1}{\cosh^2 \tau \sinh^{n-1} \tau}\, d\tau} - \frac{1}{\int_s^\infty \frac{1}{\cosh^2 \tau \sinh^{n-1} \tau}\, d\tau}$$

$$= \frac{\cosh r}{u_0(r)} - \frac{\cosh s}{u_0(s)},$$

thus

$$v(s) - \lim_{r\to\infty} v(r)$$

$$= \int_s^\infty \left(\frac{\cosh r}{u_0(r)} - \frac{\cosh s}{u_0(s)}\right) u_0(r) \sinh^{n-1} r \left(f(u(r,\theta)) - \sinh^{-2} r\Delta^\sigma u(r,\theta)\right)\, dr.$$

From (2) we have

$$\frac{(n-2)\omega_{n-1}}{2} m^{U^\kappa b} \le \lim_{r\to\infty} \int_{S^{n-1}} v(r)\, d\mu^\sigma,$$

so when we integrate over S^{n-1} we arrive at

$$\int_{S^{n-1}} v(s)\,d\mu^\sigma - \frac{(n-2)\omega_{n-1}}{2}m^{U^\kappa b}$$
$$\geq \int_s^\infty \left(\frac{\cosh r}{u_0(r)} - \frac{\cosh s}{u_0(s)}\right) u_0(r)\sinh^{n-1} r \left(\int_{S^{n-1}} f(u(r,\theta))\,d\mu^\sigma\right) dr$$
$$\geq \int_s^\infty \left(1 - \frac{\cosh s}{\cosh r}\frac{u_0(r)}{u_0(s)}\right)\cosh r \sinh^{n-1} r \left(\int_{S^{n-1}} f(u(r,\theta))\,d\mu^\sigma\right) dr,$$

with equality if the coordinates at infinity are balanced. $\square$

Proof of Proposition 3.6. The existence of the function w is guaranteed by [3, Theorem 1.2] which says that any asymptotically hyperbolic manifold is conformally related to one with scalar curvature $-n(n-1)$. The function w is a solution of the Yamabe equation

$$-\frac{4(n-1)}{n-2}\Delta^g w + \mathrm{Scal}^g w = -n(n-1)w^{\kappa+1}. \tag{8}$$

Since $\mathrm{Scal}^g \geq -n(n-1)$, the constant function 1 is a supersolution of (8). Applying the maximum principle as in the proof of Lemma 3.2 tells us that $w \leq 1$. Consequently, since both U and $\widetilde{U}$ satisfy the Yamabe equation (3), it follows from the proof of Lemma 3.2 that $U_- \leq \widetilde{U} \leq U \leq U_+$ on $\mathbb{H}^n \setminus B_{R_0}$. We set $\widetilde{u} = \widetilde{U} - 1$, $\widetilde{v} = u_0^{-1}\widetilde{u}$, and we note that $\widetilde{u} \leq u$ and $\widetilde{v} \leq v$. Since Φ are balanced coordinates at infinity for g (but not necessarily for $\widetilde{g}$) we see from (2) that

$$m^g - m^{\widetilde{g}} \geq \frac{2}{(n-2)\omega_{n-1}}\int_{S^{n-1}}(v - \widetilde{v})\,d\mu^\sigma \geq 0.$$

Again since Φ are balanced coordinates at infinity for g we conclude from Lemma 3.7 that

$$\int_{S^{n-1}}(v(s) - \widetilde{v}(s))\,d\mu^\sigma \leq \frac{(n-2)\omega_{n-1}}{2}\left(m^g - m^{\widetilde{g}}\right)$$
$$+ \int_s^\infty \left(1 - \frac{\cosh s}{\cosh r}\frac{u_0(r)}{u_0(s)}\right)\cosh r \sinh^{n-1} r \left(\int_{S^{n-1}}(f(u(r,\theta)) - f(\widetilde{u}(r,\theta)))\,d\mu^\sigma\right) dr.$$

Observe that

$$0 \leq \frac{\cosh s}{\cosh r}\frac{u_0(r)}{u_0(s)} \leq 1.$$

Moreover, recall that $u_+ = U_+ - 1 > 0$. Therefore we can use mean value theorem to show that

$$f(u) - f(\widetilde{u}) = f'(tu + (1-t)\widetilde{u})(u - \widetilde{u})$$
$$\leq \widetilde{C}(tu + (1-t)\widetilde{u})(u - \widetilde{u})$$
$$\leq \widetilde{C}u_+(u - \widetilde{u})$$
$$= \widetilde{C}u_0 v_+(u_0 v - u_0\widetilde{v})$$
$$= \widetilde{C}u_0^2 v_+(v - \widetilde{v}).$$

where $0 \leq t \leq 1$, the constant $\widetilde{C} > 0$ depends only on f, and $v_+ = u_0^{-1}u_+$. Consequently, we can estimate

$$\int_{S^{n-1}} (v(s,\theta) - \widetilde{v}(s,\theta))\, d\mu^\sigma \leq \frac{(n-2)\omega_{n-1}}{2} \left(m^g - m^{\widetilde{g}}\right)$$
$$+ \int_s^\infty F(r) \left(\int_{S^{n-1}} (v(r,\theta) - \widetilde{v}(r,\theta))\, d\mu^\sigma\right) dr,$$

where $F(r) := C' \cosh r \sinh^{n-1} r\, u_0^2(r) v_+(r)$.

We now argue as in the proof of Gronwall's lemma and prove the estimate

$$\int_{S^n} (v(s,\theta) - \widetilde{v}(s,\theta))\, d\mu^\sigma \leq \frac{(n-2)\omega_{n-1}}{2} \left(m^g - m^{\widetilde{g}}\right) e^{\int_s^\infty F(t)\, dt}. \tag{9}$$

We first consider the case when $m^g - m^{\widetilde{g}} > 0$ and set

$$G(s) := \frac{(n-2)\omega_{n-1}}{2} \left(m^g - m^{\widetilde{g}}\right) + \int_s^\infty F(r) \left(\int_{S^{n-1}} (v(r,\theta) - \widetilde{v}(r,\theta))\, d\mu^\sigma\right) dr.$$

Thus we have $\int_{S^n} (v(s,\theta) - \widetilde{v}(s,\theta))\, d\mu^\sigma \leq G(s)$, and $G(s) \geq C \left(m^g - m^{\widetilde{g}}\right)$. It is also clear that

$$G'(s) = -F(s) \int_{S^{n-1}} (v(s,\theta) - \widetilde{v}(s,\theta))\, d\mu^\sigma \geq -F(s)G(s).$$

Since $G(s) > 0$ we conclude that

$$\frac{G'(s)}{G(s)} \geq -F(s).$$

Integrating this inequality from s to ∞ we get $\ln\left(\frac{(n-2)\omega_{n-1}}{2}\left(m^g - m^{\widetilde{g}}\right)\right) - \ln G(s) \geq -\int_s^\infty F(t)\, dt$. This yields

$$G(s) \leq \frac{(n-2)\omega_{n-1}}{2} \left(m^g - m^{\widetilde{g}}\right) e^{\int_s^\infty F(t)\, dt},$$

which in its turn implies (9). Note that (9) also holds for $m^g - m^{\widetilde{g}} = 0$ which follows by passing to the limit when $m^g - m^{\widetilde{g}} > 0$ and $m^g - m^{\widetilde{g}} \to 0$ in (9).

As a consequence we can estimate the L^p-norm of $v - \widetilde{v}$ over the annulus A_{r_1,r_2} where $R_0 < r_1 < R_1 < r_2$,

$$\|v - \widetilde{v}\|^p_{L^p(A_{r_1,r_2})} = \int_{A_{r_1,r_2}} (v - \widetilde{v})^p\, d\mu^b$$
$$\leq \int_{A_{r_1,r_2}} (2v_+)^{p-1}(v - \widetilde{v})\, d\mu^b$$
$$= \int_{r_1}^{r_2} (2v_+)^{p-1} \sinh^{n-1} r \left(\int_{S^{n-1}} (v(r,\theta) - \widetilde{v}(r,\theta))\, d\mu^\sigma\right) dr$$
$$\leq C \left(m^g - m^{\widetilde{g}}\right).$$

We are now about to obtain the estimate stated in the lemma. The equation for $U - \widetilde{U}$ reads

$$-\frac{4(n-1)}{n-2}\Delta^b(U - \widetilde{U}) - n(n-1)\left(U - \widetilde{U}\right) = -n(n-1)\left(U^{\kappa+1} - \widetilde{U}^{\kappa+1}\right).$$

Since u_0 is bounded we have that

$$\|U - \widetilde{U}\|_{L^p(A_{r_1,r_2})} = \|u - \widetilde{u}\|_{L^p(A_{r_1,r_2})} \leq C \left(m^g - m^{\widetilde{g}}\right)^{1/p}.$$

By the mean value theorem we have

$$U^{\kappa+1} - \widetilde{U}^{\kappa+1} = (\kappa + 1)\left(tU + (1-t)\widetilde{U}\right)^{\kappa}\left(U - \widetilde{U}\right)$$
$$\leq CU_+^{\kappa}\left(U - \widetilde{U}\right)$$
$$\leq C\left(U - \widetilde{U}\right)$$

on A_{r_1,r_2}. Hence

$$\|U^{\kappa+1} - \widetilde{U}^{\kappa+1}\|_{L^p(A_{r_1,r_2})} \leq C\left(m^g - m^{\widetilde{g}}\right)^{1/p}.$$

Now elliptic regularity yields

$$\left\|U - \widetilde{U}\right\|_{W^{2,p}(A_{r_1',R_1})} \leq C\left(m^g - m^{\widetilde{g}}\right)^{1/p},$$

where $r_1 < r_1' < R_1$, and by embedding theorems we conclude that

$$\sup_{A_{r_1',R_1}}\left|U - \widetilde{U}\right| \leq C\left(m^g - m^{\widetilde{g}}\right)^{1/p}.$$

Set $\varphi := \log U$ and $\widetilde{\varphi} := \log \widetilde{U}$. Then $\varphi - \widetilde{\varphi}$ is non-negative, tends to zero at infinity, and satisfies

$$-\frac{4(n-1)}{n-2}\left(\Delta^b(\varphi - \widetilde{\varphi}) + \langle d(\varphi - \widetilde{\varphi}), d(\varphi + \widetilde{\varphi})\rangle_b\right) + n(n-1)\left(e^{\kappa\varphi} - e^{\kappa\widetilde{\varphi}}\right) = 0.$$

If the maximum of $\varphi - \widetilde{\varphi}$ is attained at an interior point of $\mathbb{H}^n \setminus B_{R_1}$ we get a contradiction, and thus

$$\log U - \log \widetilde{U} \leq \sup_{\partial B_{R_1}}\left(\log U - \log \widetilde{U}\right)$$

on $\mathbb{H}^n \setminus B_{R_1}$. By the mean value theorem we have

$$\log U - \log \widetilde{U} = \frac{U - \widetilde{U}}{tU + (1-t)\widetilde{U}} \begin{cases} \geq \dfrac{U - \widetilde{U}}{U_+(R_1)} \\ \leq \dfrac{U - \widetilde{U}}{U_-(R_1)} \end{cases}.$$

Thus

$$U - \widetilde{U} \leq U_+(R_1)\left(\log U - \log \widetilde{U}\right)$$
$$\leq U_+(R_1)\sup_{\partial B_{R_1}}\left(\log U - \log \widetilde{U}\right)$$
$$\leq \frac{U_+(R_1)}{U_-(R_1)}\sup_{\partial B_{R_1}}\left(U - \widetilde{U}\right)$$
$$\leq C\left(m^g - m^{\widetilde{g}}\right)^{1/p}$$

on $\mathbb{H}^n \setminus B_{R_1}$, which concludes the proof of the lemma. $\qquad\square$

3.3. **Strategy of the proof of Theorem A for metrics with constant scalar curvature.** In this subsection, we discuss the main strategy of the proof of Theorem A for metrics with constant scalar curvature, deferring the proof of technical lemmas to the next subsection. The basic idea is to apply the positive mass theorem to a certain 1-parameter family of metrics. To define it, we first modify the metric g in an annulus (see Equation (11)) and conformally transform it to fulfill the assumption $\mathrm{Scal} \geq -n(n-1)$ of the positive mass theorem.

In the first lemma we prove the existence of a function V which solves $\Delta^g V = nV$ and is asymptotic to $V_{(0)}$. For functions V_1 and V_2 on M we write $V_1 \sim V_2$ if V_1/V_2 tends to 1 at infinity.

Let R_0', R_0'', R_1' and R_1'' be constants such that

$$R_0 < R_0' < R_0'' < R_1 < R_1' < R_1''.$$

Lemma 3.8. *Let* $(M, g, \Phi, U) \in \mathcal{A}$. *There exists a unique solution* V^g *to the equation*

$$\Delta^g V = nV \tag{10}$$

such that $V^g \sim V_{(0)}$. *There exist universal functions*

$$V_\pm : \mathbb{H}^n \setminus B_{R_0'} \to \mathbb{R}$$

such that for some constants $C_0, C_1 > 0$,

$$\left| V_\pm - V_{(0)} \right| \leq C_0 e^{-(n-1)r},$$
$$V_- \leq V^g \leq V_+,$$

and

$$\left| dV^g - dV_{(0)} \right|_g \leq C_1 e^{-(n-1)r}$$

on $\mathbb{H}^n \setminus B_{R_0'}$. *Further, there exist constants* $B_2, B_3, \ldots$ *depending only on* R_0', R_0'' *and* R_1'' *such that for any integer* $k \geq 2$

$$\left| \nabla^{(k)} V^g \right| \leq B_k \ \text{on} \ A_{R_0'', R_1''}.$$

Define

$$T := \widehat{\mathrm{Ric}}^g - \frac{\overset{\circ}{\mathrm{Hess}}(V^g)}{V^g}.$$

We choose a smooth function χ such that

$$\chi = \begin{cases} 0 & \text{on } B_{R_0''}, \\ 1 & \text{on } A_{R_1, R_1'}, \\ 0 & \text{on } \mathbb{H}^n \setminus B_{R_1''}. \end{cases}$$

and define the metric

$$g_s := g - s\chi T \tag{11}$$

for small values of the parameter s.

Next we recall the definition of the weighted local Sobolev spaces, see [14] for more details on these spaces. Let $p \in (1, \infty)$, k a non negative integer, and $\delta \in \mathbb{R}$ be given. We define the $X_\delta^{k,p}(M, \mathbb{R})$ function space as the set of functions $u \in W_{loc}^{k,p}(M, \mathbb{R})$ such that the norm

$$\|u\|_{X_\delta^{k,p}(M,\mathbb{R})} = \sup_{x \in M} e^{\delta r} \|u\|_{W^{k,p}(B_1(x),\mathbb{R})}$$

is finite. This space is a Banach space.

We will conformally transform the metrics g_s to have constant scalar curvature $\mathrm{Scal} = -n(n-1)$. The details of this are taken care of by the following lemma.

Lemma 3.9. *There exists $s_0 > 0$ such that for all $s \in [-s_0, s_0]$ and any $(M, g, \Phi, U) \in \mathcal{A}_0$ it holds that*

$$\frac{1}{2}g \leq g_s \leq 2g$$

and

$$|\mathrm{Scal}^{g_s} + n(n-1)| \leq n - 1.$$

For any $s \in [-s_0, s_0]$ there exists a unique positive function φ_s on M which is bounded from above and away from zero such that the metric

$$\lambda_s := \varphi_s^\kappa g_s,$$

has constant scalar curvature $-n(n-1)$. The function φ_s satisfies

$$\left(\frac{n-1}{n}\right)^{1/\kappa} \leq \varphi_s \leq \left(\frac{n+1}{n}\right)^{1/\kappa}.$$

There are constants $C_0, C_1, \ldots$ such that

$$\left|\nabla^{(k)}(\varphi_s - 1)\right| \leq C_k e^{-nr} \tag{12}$$

holds on $\mathbb{H}^n \setminus B_{R_1}$ for all $k \geq 0$. Further, the map $s \mapsto \varphi_s - 1$ from the interval $[-s_0, s_0]$ to $X_\delta^{2,p}(M, g)$ is analytic for any $p \in (n, \infty)$ and $\delta \in \left(\frac{n}{2}, n\right)$.

For $V = V_{(0)} = \cosh r$ we set $H(s) := H_\Phi^{\lambda_s}(V)$. This is the time component of the mass functional, which gives an upper bound on the mass, $m^{\lambda_s} \leq \frac{1}{2(n-1)\omega_{n-1}} H(s)$. Since the coordinates at infinity are balanced for g we have $m^g = m^{\lambda_0} = \frac{1}{2(n-1)\omega_{n-1}} H(0)$. In what follows we will denote derivatives with respect to the parameter s by a dot.

Lemma 3.10. *The map $s \mapsto H(s)$ is a C^2 function. Further, there is a constant Λ independent of $(M, g, \Phi, U) \in \mathcal{A}_0$ such that*

$$|\ddot{H}(s)| \leq \Lambda.$$

In the next proposition we find that $\dot{H}(0)$ is related to the L^2-norm of $\widehat{\mathrm{Ric}}^g - \frac{1}{V^g}\overset{\circ}{\mathrm{Hess}}V^g$ on an annulus, which can be interpreted as a measure of "non-staticity" of the metric g on the annulus.

Proposition 3.11. *Suppose $(M, g, \Phi, U) \in \mathcal{A}_0$ and $H(s)$ is defined as above, then*

$$\dot{H}(0) = \int_M \chi V^g \left|\widehat{\mathrm{Ric}}^g - \frac{\overset{\circ}{\mathrm{Hess}}V^g}{V^g}\right|_g^2 d\mu^g.$$

We are now ready to prove Theorem A.

Proof of Theorem A. Applying Taylor's formula to $H(s)$ on the interval $(-s_0, s_0)$ we find

$$H(s) = H(0) + s\dot{H}(0) + \int_0^s (s-t)\ddot{H}(t)dt$$

$$\leq H(0) + s\dot{H}(0) + \Lambda \int_0^s (s-t)dt$$

$$\leq H(0) + s\dot{H}(0) + \frac{\Lambda}{2}s^2.$$

From the assumption that the positive mass theorem holds for any asymptotically hyperbolic metric on M we have $H(s) \geq 2(n-1)\omega_{n-1}m^{\lambda_s} \geq 0$ for $s \in (-s_0, s_0)$. That is

$$0 \leq H(0) + s\dot{H}(0) + \frac{\Lambda}{2}s^2.$$

Assuming that $H(0) \leq \frac{2s_0^2}{\Lambda}$, we write the previous inequality with $s = -\sqrt{\frac{2H(0)}{\Lambda}}$ and get

$$\dot{H}(0) \leq \sqrt{2\Lambda H(0)} = \sqrt{4\Lambda(n-1)\omega_{n-1}m^g}. \tag{13}$$

Let ε be an arbitrary positive number. We claim that there exists $\delta > 0$ such that any (M, g, Φ, U) belonging to $\mathcal{A}_0$ and having mass $m^g \leq \delta$ satisfies

$$\sup_{\mathbb{H}^n \setminus B_{R_1}} |U - 1| \leq \varepsilon.$$

To prove this we argue by contradiction and assume that there is a sequence (M_k, g_k, Φ_k, U_k) of elements of $\mathcal{A}_0$ such that the mass $m_k := m^{g_k}$ tends to zero while $|U_k - 1| \geq \varepsilon$. Using Lemmas 3.2, 3.8 and [14, Proposition 2.3] (Rellich theorems for weighted local Sobolev spaces), we construct functions U_∞ and V_∞ on $\mathbb{H}^n \setminus B_{R_0'}$ as limits of some subsequence of U_k and V^{g_k} as follows. Choose arbitrary $p \in (n, \infty)$ and $\delta \in \left(\frac{n}{2}, n\right)$.

- From Lemma 3.2, the sequence $U_k - 1$ is bounded in $X_n^{3,p}(\mathbb{H}^n \setminus B_{R_0'})$. Hence there exists a subsequence converging to some $U_\infty - 1$ in $X_\delta^{2,p}$.
- For V, it suffices to remark that the sequence V_k is uniformly bounded in $W^{3,p}(K)$ for any compact subset $K \subset \mathbb{H}^n \setminus \mathring{B}_{R_0'}$ by standard elliptic regularity. Hence, by a diagonal process, we can construct a subsequence of functions V_k converging in the $W^{2,p}$-norm on any compact subset.

The function U_∞ solves (3) and V_∞ solves $\Delta^{g_\infty}V_\infty = nV_\infty$ where $g_\infty := U_\infty^\kappa b$. They satisfy the asymptotics of Lemmas 3.2 and 3.8. Further,

$$\sup_{\mathbb{H}^n \setminus B_{R_1}} |U_\infty - 1| \geq \varepsilon. \tag{14}$$

The metric g_∞ has mass zero since the mass depends continuously on $U - 1 \in X_\delta^{2,p}$ (see the proof of Lemma 3.10).

Proposition 3.11 together with the estimate (13) applied to (M_k, g_k, Φ_k, U_k) gives the inequality

$$\int_M \chi V^{g_k} \left| \widehat{\mathrm{Ric}}^{g_k} - \frac{\mathring{\mathrm{Hess}}V^{g_k}}{V^{g_k}} \right|_{g_k}^2 d\mu^{g_k} \leq \sqrt{4\Lambda(n-1)\omega_{n-1}m_k}$$

for any k. In particular we obtain

$$\int_{\mathbb{H}^n \setminus B_{R_0'}} \chi V^{g_\infty} \left| \widehat{\mathrm{Ric}}^{g_\infty} - \frac{\mathring{\mathrm{Hess}}V^{g_\infty}}{V^{g_\infty}} \right|_{g_\infty}^2 d\mu^{g_\infty} = 0$$

when we let k tend to infinity. Therefore

$$\widehat{\mathrm{Ric}}^{g_\infty} = \frac{\mathring{\mathrm{Hess}}V^{g_\infty}}{V^{g_\infty}}$$

on $A_{R_0'', R_1'}$. By analyticity this equality holds on all of $\mathbb{H}^n \setminus B_{R_0'}$. From Proposition A.4 and the fact that the metric g_∞ has zero mass, we conclude that g_∞ is hyperbolic. This forces $U_\infty = 1$ which contradicts (14), we have thus proved the claim stated above.

The proof of Theorem A is then concluded by Proposition 3.6 followed by Proposition 3.4. $\qquad\square$

3.4. Proof of lemmas.

We now complete the proof of Theorem A by proving the lemmas stated in the previous subsection.

Proof of Lemma 3.8. We first construct $V_\pm$. The construction being lengthy, we give only the argument for V_+. We want V_+ to be a supersolution for Equation 10,

$$-\Delta^g V_+ + nV_+ \geq 0.$$

Since $g = U^\kappa b$ on $\mathbb{H}^n \setminus B_{R_0}(0)$ the previous inequality is equivalent to

$$-\Delta^b V_+ - 2\left\langle \frac{dU}{U}, dV_+ \right\rangle + nU^\kappa V_+ \geq 0.$$

We choose V_+ to be a function of r so

$$-V_+'' - (n-1)\coth(r)V_+' - 2\frac{\partial_r U}{U}V_+' + nU^\kappa V_+ \geq 0.$$

where a prime denotes a derivative with respect to r. From Lemma 3.2, there exists a universal constant A_1' depending only on R_0' such that $\frac{\partial_r U}{U} \leq A_1 e^{-nr}$. Assuming that $V_+, V_+' \geq 0$, the previous inequality will be satisfied provided that

$$- V_+'' - (n-1)\coth(r)V_+' - 2A_1' e^{-nr}V_+' + n\varphi_-^\kappa V_+ = 0, \tag{15}$$

where φ_- is the Anti-de Sitter-Schwarzschild solution vanishing at $r = R_0$. Let λ be a positive real number to be chosen later. From standard theory, there exists a unique solution to Equation (15) defined on $[R_0', \infty)$ such that $V_+(R_0') = \lambda$ and $V_+'(R_0') = 0$.

We first claim that V_+ and V_+' are both positive functions on (R_0', ∞). Indeed, rewriting Equation (15) as

$$V_+'' + \left((n-1)\coth(r) + 2A_1' e^{-nr}\right) V_+' = n\varphi_-^\kappa V_+, \tag{16}$$

setting $R := \inf\{r > R_0', V_+(r) \leq 0\}$, and assuming that $R < \infty$, we have $V_+ > 0$ on (R_0', R) and $V_+(R) = 0$. Hence, regarding (16) as a first order homogeneous ordinary differential equation for V_+', we conclude that $V_+' > 0$ on (R_0', R). In particular, $V_+(R) \geq V_+(R_0') = \lambda > 0$. This contradicts the definition of R. The claim is proved.

Next we prove that $V_+ = \alpha\lambda\cosh(r) + O(e^{-(n-1)r})$ for some constant $\alpha > 0$. Hence setting $\lambda = 1/\alpha$, we get a supersolution to Equation (10) such that $V_+ \sim \cosh(r) = V_{(0)}$. To prove this second claim we set $V_+(r) := \cosh(r)v_+(r)$. By a straightforward calculation, we find that V_+ satisfies (15) if and only if v_+ satisfies

$$v_+'' + \left(2\tanh(r) + (n-1)\coth(r) + 2A_1' e^{-nr}\right) v_+'$$
$$+ \left(2A_1' e^{-nr}\tanh(r) + n\left(\varphi_-^\kappa - 1\right)\right) v_+ = 0.$$

From the first claim we have $v_+ > 0$. We introduce $k_+ := \frac{v'_+}{v_+}$ and obtain the following Riccati equation for k_+,

$$k'_+ + k_+^2 + \left(2\tanh(r) + (n-1)\coth(r) + 2A'_1 e^{-nr}\right) k_+$$
$$+ \left(2A'_1 e^{-nr}\tanh(r) + n\left(\varphi_-^\kappa - 1\right)\right) = 0. \tag{17}$$

Without loss of generality, we can assume that A'_1 is chosen so large that

$$2A'_1 e^{-nr}\tanh(r) + n\left(\varphi_-^\kappa - 1\right) \geq 0$$

on (R'_0, ∞). From the boundary condition $V'_+(R'_0) = 0$ we have

$$v'_+(R'_0) = -\tanh(R'_0)v_+(R'_0) < 0.$$

It is then fairly straightforward to argue that $-1 < k_+ < 0$ on (R'_0, ∞). For this let R be the smallest $r > R'_0$ such that $k_+(r) \geq 0$. Then $k_+(R) = 0$ and, from Equation (17), $k'_+(R) < 0$ so $k_+(r) > 0$ for some r slightly smaller than R, contradicting the definition of R. This estimate can be further refined. We select $\alpha \in \left(\frac{n}{2}, n\right)$ and set $k_+^- := -e^{-\alpha(r-r_0)}$ for some r_0 to be chosen later. Then $k_+^- \geq -1$ on the interval $[r_0, \infty)$. Hence

$$(k_+^-)' + (k_+^-)^2 + \left(2\tanh(r) + (n-1)\coth(r) + 2A'_1 e^{-nr}\right) k_+^-$$
$$= \left(\alpha + e^{-\alpha(r-r_0)} - 2\tanh(r) - (n-1)\coth(r) - 2A'_1 e^{-nr}\right) e^{-n(r-r_0)}$$
$$\leq (\alpha - n)\, e^{-\alpha(r-r_0)},$$

where we used the inequality

$$2\tanh(r) + (n-1)\coth(r) = 2\left(\frac{1}{\coth(r)} + \coth(r)\right) + (n-3)\coth(r)$$
$$\geq 2 + (n-3)\coth(r)$$
$$\geq n - 1.$$

So, choosing r_0 large enough, we can ensure that

$$(k_+^-)' + (k_+^-)^2 + \left(2\tanh(r) + (n-1)\coth(r) + 2A'_1 e^{-nr}\right) k_+^-$$
$$+ \left(2A'_1 e^{-nr}\tanh(r) + n\left(\varphi_-^\kappa - 1\right)\right) < 0$$

on the interval $[r_0, \infty)$. A slight modification of the previous argument shows that $k_+^- \leq k_+ \leq 0$. Equation (17) then implies $k'_+ = O(e^{-nr})$. Together with the fact that $k_+ \to 0$ at infinity, this implies $k_+ = O(e^{-nr})$. Thus we infer that

$$\log v_+(r) = \log \lambda + \mu + O(e^{-nr})$$

for some constant μ. Hence,

$$v_+(r) = \lambda e^\mu + O(e^{-nr}).$$

This proves the second claim with $\alpha = e^\mu$.

Finally remark that since $k_+(r) = \frac{v'_+(r)}{v_+(r)} \leq 0$ and $v_+ \to 1$ at infinity, $v_+(r) \geq 1$ so $V_+ \geq V_{(0)}$.

The construction of the subsolution V_- on $\mathbb{H}^n \setminus B_{R'_0}$ is entirely similar. The only difference is that we select $V_-(R'_0) = 0$ and $V'_-(R'_0) > 0$. The function V_- then satisfies $V_- \leq V_{(0)}$.

From now on we will work on the entire manifold M. Using the diffeomorphism Φ we define open sets B'_R in M through the relation $\Phi(M \setminus B'_R) = \mathbb{H}^n \setminus B_R$ for

$R \geq R_0$. The set B'_R is the part of M inside an approximate geodesic sphere in the asymptotically hyperbolic end. By abusing notation we consider the functions $V_{\pm}$ and $V_{(0)}$ as defined on $M \setminus K$ through the diffeomorphism Φ.

Our proof of existence of the function V^g follows [16]. For any $r > R'_0$ there exists a unique function V^r solving (10) inside the sphere of radius r with Dirichlet data $V = V_{(0)}$ on $\partial B'_r$. From the maximum principle, $V^r \geq 0$. Then a second application of the maximum principle in the annulus $\overline{B'}_r \setminus B'_{R'_0}$ yields $V^r \geq V_-$. We extend the function V_+ by λ on $B'_{R'_0}$. This new function V_+ is a C^1-supersolution of (10) in the weak sense. Hence $V^r \leq V_+$ (see for example [15, Theorem 8.1] for more details). In particular, the functions V^r are uniformly bounded on compact subsets. Then a standard argument using elliptic regularity and a diagonal extraction process yields existence of the function V^g. Similarly, we extend the function V_- by zero on $B'_{R'_0}$. The function V_- extended this way becomes a subsolution in the weak sense so the functions V^r satisfy $V^r \geq V_-$. In the limit, the function V^g is pinched between V_- and V_+,

$$V_- \leq V^g \leq V_+.$$

This proves that $V^g - V_{(0)} = O(e^{-(n-1)r})$.

We note that

$$\Delta^g V_{(0)} = U^{-\kappa}(n \cosh(r) + 2\frac{\partial_r U}{U} \sinh(r) = nV_{(0)} + O(e^{-(n-1)r}).$$

Hence,

$$(-\Delta^g + n)\left(V^g - V_{(0)}\right) = O(e^{-(n-1)r}).$$

The estimates for $d(V^g - V_{(0)})$ and $\left|\nabla^{(k)} V^g\right|$ follow from standard elliptic regularity.

We finally prove uniqueness of V^g. Assume that V_1 is the function we constructed before so that $V_- \leq V_1 \leq V_+$ and V_2 is another function satisfying $\Delta V_2 = nV_2$, $V_2 \sim V_{(0)}$. From the strong maximum principle we have $V_1 > 0$. We compute

$$nV_2 = \Delta\left(\frac{V_2}{V_1}V_1\right)$$

$$= V_1\Delta\frac{V_2}{V_1} + 2\left\langle dV_1, d\left(\frac{V_2}{V_1}\right)\right\rangle + \frac{V_2}{V_1}\Delta V_1$$

$$= V_1\Delta\frac{V_2}{V_1} + 2\left\langle dV_1, d\left(\frac{V_2}{V_1}\right)\right\rangle + nV_2,$$

so

$$0 = \Delta\frac{V_2}{V_1} + 2\left\langle \frac{dV_1}{V_1}, d\left(\frac{V_2}{V_1}\right)\right\rangle.$$

Since $V_1 \sim V_2$, the function V_2/V_1 tends to 1 at infinity. From the strong maximum principle (which can be applied here since if V_2/V_1 is not constant, the maximum of $|V_2/V_1|$ is attained at some point $p \in M$), we conclude that $V_2/V_1 = 1$. $\qquad \square$

Proof of Lemma 3.9. From Lemmas 3.2 and 3.8, there exist universal constants $B_0, B_1, \ldots$ such that

$$\left|\nabla^{(k)}T\right| \leq B_k$$

for any $k = 0, 1, \ldots$ on the support of χ. Hence

$$|g(X, X) - g_s(X, X)| = |sT(X, X)| \leq |s|B_0 g(X, X).$$

for any $X \in TM$. So if $|s| \leq \frac{1}{2B_0}$ we have

$$\frac{1}{2}g(X,X) \leq g_s(X,X) \leq \frac{3}{2}g(X,X).$$

We denote by ∇^{g_s} the Levi-Civita connection of g_s. The difference between ∇^{g_s} and ∇^{g_0} is a symmetric vector valued 2-tensor $\Gamma(s)$,

$$\nabla_X^{g_s} Y - \nabla_X^{g_0} Y = \Gamma(s)(X,Y).$$

In coordinates $\Gamma(s)$ is given by

$$\Gamma_{ij}^k(s) = \frac{1}{2}g_s^{kl}\left(\nabla_i(g_s)_{lj} + \nabla_j(g_s)_{il} - \nabla_l(g_s)_{ij}\right)$$

$$= -\frac{s}{2}g_s^{kl}\left(\nabla_i(\chi T_{lj}) + \nabla_j(\chi T_{il}) - \nabla_l(\chi T_{ij})\right),$$

where we have denoted by $\nabla = \nabla^{g_0}$ the Levi-Civita connection of the metric $g_0 = g$. The scalar curvature of the metric g_s can be written as follows,

$$\mathrm{Scal}^{g_s} = g_s^{ij}\mathrm{Ric}_{ij}^{g_0} + g_s^{jl}\left(\nabla_i\Gamma_{jl}^i(s) - \nabla_l\Gamma_{ij}^i(s) + \Gamma_{ip}^i(s)\Gamma_{jl}^p(s) - \Gamma_{lp}^i(s)\Gamma_{ij}^p(s)\right).$$

From this formula it is not complicated to see that there is a constant $s_0 > 0$, $s_0 \leq \frac{1}{2B_0}$, depending only on B_0, B_1, B_2 and n such that

$$|\mathrm{Scal}^{g_s} - \mathrm{Scal}^g| \leq n - 1$$

for $|s| \leq s_0$. From the bound on Scal^{g_s} it follows that the constant functions $\varphi_- = \left(\frac{n-1}{n}\right)^{1/\kappa}$ and $\varphi_+ = \left(\frac{n+1}{n}\right)^{1/\kappa}$ are respectively a sub-solution and a super-solution of the Yamabe equation

$$-\frac{4(n-1)}{n-2}\Delta^{g_s}\varphi_s + \mathrm{Scal}^{g_s}\varphi_s + n(n-1)\varphi_s^{\kappa+1} = 0. \tag{18}$$

Arguing as in the proof of Proposition 3.6 there exists a unique solution φ_s of (18) such that φ_s is bounded from above and away from zero. Further $\varphi_- \leq \varphi_s \leq \varphi_+$.

We next prove that the map $s \mapsto \varphi_s$ is analytic. We consider the map

$$\begin{array}{rcl}\Xi: \ \Omega \times [-s_0, s_0] & \to & X_\delta^{0,p} \\ (u,s) & \mapsto & -\frac{4(n-1)}{n-2}\Delta^{g_s}u + \mathrm{Scal}^{g_s}(u+1) + n(n-1)(u+1)^{\kappa+1},\end{array}$$

where $\Omega = \{u \in X_\delta^{2,p}, u > -1\}$. Hence, for any $s \in [-s_0, s_0]$, $u_s = \varphi_s - 1$ is the only solution to the equation $\Xi(u,s) = 0$. Further Ξ is an analytic function. The differential of Ξ with respect to u at any point (u_s, s) is given by

$$\begin{array}{rcl}D_u\Xi(u_s,s): \ X_\delta^{2,p} & \to & X_\delta^{0,p} \\ v & \mapsto & -\frac{4(n-1)}{n-2}\Delta^{g_s}v + \left(\mathrm{Scal}^{g_s} + (\kappa+1)n(n-1)\varphi_s^\kappa\right)v.\end{array}$$

We remark that

$$\mathrm{Scal}^{g_s} + (\kappa+1)n(n-1)\varphi_s^\kappa \geq -(n+1)(n-1) + (\kappa+1)(n-1)^2$$

$$\geq \frac{2n(n-1)}{n-2},$$

from which it follows that the L^2-kernel of $D_u\Xi(u_s,s)$ is zero. From the Fredholm alternative (see [14, Proof of Proposition 5.1]), we conclude that $D_u\Xi(u_s,s)$ is invertible. Using the implicit function theorem, this proves that the map $s \mapsto \varphi_s - 1 \in X_\delta^{2,p}$ is C^2.

To prove the asymptotics of φ_s, remark that the metric λ_s falls into the class $\mathcal{A}(R_2)$. Hence the estimates (12) are consequences of Lemma 3.2. $\qquad\square$

Proof of Lemma 3.10. We first estimate the first and second derivatives of φ_s with respect to s. We differentiate Equation (18) with respect to s and find the following equation for $\dot{\varphi}_s$,

$$-\frac{4(n-1)}{n-2}\Delta^{g_s}\dot{\varphi}_s + \mathrm{Scal}^{g_s}\dot{\varphi}_s + (\kappa+1)n(n-1)\varphi_s^{\kappa}\dot{\varphi}_s$$
$$= \frac{4(n-1)}{n-2}\frac{\partial\Delta^{g_s}}{\partial s}\varphi_s - \frac{\partial\mathrm{Scal}^{g_s}}{\partial s}\varphi_s.$$

Note that the right hand side has support in the annulus A_{R_0',R_1} and by Lemma 3.9 it is thus bounded by some universal constant C. We also remark that, since

$$\mathrm{Scal}^{g_s} + (\kappa+1)n(n-1)\varphi_s^{\kappa} > \frac{2n(n-1)}{n-2}$$

and since $\dot{\varphi}_s$ tends to zero at infinity (this is a consequence of $\dot{\varphi}_s \in X_\delta^{2,p}$), we have

$$\sup|\dot{\varphi}_s| \le \frac{n-2}{2n(n-1)}C.$$

By standard techniques one can then prove that $\|\dot{\varphi}_s\|_{X^{2,p}(\mathbb{H}^n\setminus B_{R_0'})} \le C$ for some universal constant C. The same strategy can then be used to study the second order derivative of φ_s. However, the calculations are lengthy and we do not include the argument here.

The last step is to prove that $H(s)$ is a C^2 function of s. For this we write $H(s)$ as follows (see [18, page 114] or [11] for more details),

$$H(s) = H_\Phi^{\lambda_s}(V)$$
$$= \int_{S_{R_2}}\left(V(\mathrm{div}^b e_s - d\,\mathrm{tr}^b e_s) + (\mathrm{tr}^b e_s)dV - e_s(\nabla^b V, \cdot)\right)(\nu_{R_2})\,d\mu^b$$
$$\quad + \int_{\mathbb{H}^n\setminus B_{R_2}}\left(V\left(\mathrm{Scal}^{\lambda_s} - \mathrm{Scal}^b\right) + \mathcal{Q}(e_s, V)\right)d\mu^b$$
$$= \int_{S_{R_2}}\left(V(\mathrm{div}^b e_s - d\,\mathrm{tr}^b e_s) + (\mathrm{tr}^b e_s)dV - e_s(\nabla^b V, \cdot)\right)(\nu_{R_2})\,d\mu^b$$
$$\quad + \int_{\mathbb{H}^n\setminus B_{R_2}}\mathcal{Q}(e_s, V)\,d\mu^b$$

where $e_s = \lambda_s - b = (\varphi_s^{\kappa}U^{\kappa} - 1)b$, and $\mathcal{Q}(V, e)$ is an expression which is linear in V, quadratic in e_s and its first derivatives and cubic in $(\lambda_s)^{-1}$. It corresponds to the negative of the non-linear terms in the Taylor expansion of

$$\int_{\mathbb{H}^n\setminus B_{R_2}}V\left(\mathrm{Scal}^{\lambda_s} - \mathrm{Scal}^b\right)d\mu^b$$

with respect to $e_s = \lambda_s - b$. Since $\lambda_s = (U\varphi_s)^{\kappa}b$ on $\mathbb{H}^n\setminus B_{R_2}$, this expression can be explicitly computed,

$$\mathcal{Q}(e_s, V) = V\left((n-1)\left(\frac{1}{\psi_s^2} - 1\right)\Delta^b\psi_s + n(n-1)\frac{(\psi_s - 1)^2}{\psi_s} + \frac{(n-1)(n-6)}{4\psi_s}\left|\frac{d\psi_s}{\psi_s}\right|_b^2\right),$$

where $\psi_s := (U\varphi_s)^{\kappa}$. Written in this form, one can see that $H(s)$ depends in a C^2 fashion on s from standard theorems on the derivative of integrals.

From the estimates we have found for $\dot\varphi(s)$ and $\ddot\varphi(s)$ together with Lemmas 3.2 and 3.9 it is not complicated to deduce that $\dot H(s)$ and $\ddot H(s)$ are uniformly bounded on the interval $[-s_0, s_0]$. □

Proof of Proposition 3.11. For λ_s we have

$$e_s = \lambda_s - b = \varphi_s^\kappa\left(g - s\chi\left(\widehat{\mathrm{Ric}}^g - \frac{\mathring{\mathrm{Hess}}V^g}{V^g}\right)\right) - b,$$

the derivative of this with respect to s evaluated at $s = 0$ is

$$\dot e = \kappa\dot\varphi g - \chi\left(\widehat{\mathrm{Ric}}^g - \frac{\mathring{\mathrm{Hess}}V^g}{V^g}\right) =: e^1 + e^2.$$

The conformal factors φ_s satisfy the Yamabe equation

$$-\frac{4(n-1)}{n-2}\Delta^{g_s}\varphi_s + \mathrm{Scal}^{g_s}\varphi_s = -n(n-1)\varphi_s^{\kappa+1}.$$

Differentiating this at $s = 0$ and using the fact that $\varphi_0 = 1$ we find that

$$-\frac{4(n-1)}{n-2}\Delta^g\dot\varphi + \dot{\mathrm{Scal}}^g(\dot g) + \mathrm{Scal}^g\dot\varphi = -n(n-1)\frac{n+2}{n-2}\dot\varphi,$$

or

$$\frac{4(n-1)}{n-2}\left(\Delta^g\dot\varphi - n\dot\varphi\right) = \dot{\mathrm{Scal}}^g(\dot g). \tag{19}$$

We compute

$$\dot H(0) = \frac{d}{ds}\left(H_\Phi^{\lambda_s}(V)\right)_{|s=0}$$

$$= \lim_{r\to\infty}\int_{S_r}\left(V(\mathrm{div}^b\dot e - d\,\mathrm{tr}^b\dot e) + (\mathrm{tr}^b\dot e)dV - \dot e(\nabla^b V, \cdot)\right)(\nu_r)\,d\mu^b \tag{20}$$

$$= \lim_{r\to\infty}\int_{S_r}\left(V^g(\mathrm{div}^g\dot e - d\,\mathrm{tr}^g\dot e) + (\mathrm{tr}^g\dot e)dV^g - \dot e(\nabla^g V^g, \cdot)\right)(\nu_r^g)\,d\mu^g,$$

where we can change from the metric b to the metric g since g is asymptotically hyperbolic and the function V^g has the asymptotics specified in Lemma 3.8. Note that since e_2 has compact support its contribution to $\dot H(0)$ is zero. For the terms with $e^1 = \kappa\dot\varphi g$ in (20) we have

$$\dot H(0) = \lim_{r\to\infty}\int_{S_r}\left(V^g(\mathrm{div}^g e^1 - d\,\mathrm{tr}^g e^1) + (\mathrm{tr}^g e^1)dV^g - e^1(\nabla^g V^g, \cdot)\right)(\nu_r^g)\,d\mu^g$$

$$= \lim_{r\to\infty}\frac{4(n-1)}{(n-2)}\int_{S_r}\left(\dot\varphi dV^g - V^g d\dot\varphi\right)(\nu_r^g)\,d\mu^g$$

$$= \frac{4(n-1)}{(n-2)}\int_M \mathrm{div}^g\left(\dot\varphi dV^g - V^g d\dot\varphi\right)d\mu^g$$

$$= \frac{4(n-1)}{(n-2)}\int_M\left(V^g\Delta^g\dot\varphi - \dot\varphi\Delta^g V^g\right)d\mu^g$$

$$= \frac{4(n-1)}{(n-2)}\int_M V^g\left(\Delta^g\dot\varphi - n\dot\varphi\right)d\mu^g$$

$$= \int_M V^g\dot{\mathrm{Scal}}^g(\dot g)\,d\mu^g,$$

where the last equality was obtained using (19). Here $\dot{g} = -\chi \left(\widehat{\mathrm{Ric}}^g - \frac{\mathring{\mathrm{Hess}}V^g}{V^g} \right)$ is traceless. So from the formula for the first variation of scalar curvature, see [5, Theorem 1.174], we obtain

$$\dot{\mathrm{Scal}^g}(\dot{g}) = \mathrm{div}^g \mathrm{div}^g \dot{g} - \Delta^g \mathrm{tr}^g \dot{g} - \langle \dot{g}, \mathrm{Ric}^g \rangle_g$$

$$= \mathrm{div}^g \mathrm{div}^g \dot{g} - \langle \dot{g}, \widehat{\mathrm{Ric}}^g \rangle_g$$

$$= \chi \left| \widehat{\mathrm{Ric}}^g \right|_g^2 - \frac{\chi}{V^g} \left\langle \mathring{\mathrm{Hess}}(V^g), \widehat{\mathrm{Ric}}^g \right\rangle - \mathrm{div}^g \mathrm{div}^g (\chi \widehat{\mathrm{Ric}}^g) + \mathrm{div}^g \mathrm{div}^g \frac{\chi \mathring{\mathrm{Hess}}V^g}{V^g}.$$

Thus, replacing this expression in the formula for $\dot{m}$ and integrating by parts, we get

$$\dot{H}(0) = \int_M \chi V^g \left| \widehat{\mathrm{Ric}}^g \right|_g^2 d\mu^g - 2 \int_M \chi \left\langle \widehat{\mathrm{Ric}}^g, \mathring{\mathrm{Hess}}V^g \right\rangle_g d\mu^g$$

$$+ \int_M \chi \frac{1}{V^g} \left| \mathring{\mathrm{Hess}}V^g \right|_g^2 d\mu^g$$

$$= \int_M \chi V^g \left| \widehat{\mathrm{Ric}}^g - \frac{1}{V^g} \mathring{\mathrm{Hess}}V^g \right|_g^2 d\mu^g.$$

$\square$

4. NEAR EQUALITY CASE OF THE POSITIVE MASS THEOREM: THE SPIN CASE

In this section we will prove a version of Theorem A with an argument using spinors. This follows closely the ideas of [6, Section 12], see also the appendix of [21]. We only give a sketch of the argument. We first introduce the following class of asymptotically hyperbolic manifolds.

Definition 4.1. For $R_0 > 0$, we define the class $\mathcal{A}^{\mathrm{Spin}}(R_0)$ of 4-tuples (M, g, Φ, U) such that

- (M, g) is a complete asymptotically hyperbolic Riemannian spin manifold with respect to Φ, where Φ is a diffeomorphism from the exterior of a compact set $K \subset M$ to $\mathbb{H}^n \setminus \overline{B}_{R_0}$,
- $\mathrm{Scal}^g \geq -n(n-1)$, and $\mathrm{Scal}^g = -n(n-1)$ on $M \setminus K$,
- U is a positive function on $\mathbb{H}^n \setminus B_{R_0}$ such that $U \to 1$ infinity and $\Phi_* g = U^\kappa b$,
- the coordinates at infinity Φ are balanced.

We prove the following theorem on the near-equality case of the positive mass theorem for spin manifolds.

Theorem B. *Let $R_1 > R_0$ and $\varepsilon > 0$. Then there is a constant $\delta > 0$ so that*

$$|U - 1| \leq \varepsilon e^{-nr}$$

on $\mathbb{H}^n \setminus B_{R_1}$ for all $(M, g, \Phi, U) \in \mathcal{A}^{\mathrm{Spin}}(R_0)$ with $m^g < \delta$.

We fix the constant $R_0 > 0$ and abbreviate $\mathcal{A}^{\mathrm{Spin}}(R_0) = \mathcal{A}^{\mathrm{Spin}}$. We begin by describing the relationship between Killing spinors and the asymptotically hyperbolic mass, for this we follow closely the discussion in [11, Section 4].

Since M is a spin manifold there is a spin structure and an associated spinor bundle ΣM on (M, g). On ΣM we define the connection $\widehat{\nabla}$ by

$$\widehat{\nabla}_X^g \varphi := \nabla_X^g \varphi + \frac{i}{2} X \cdot \varphi.$$

Here ∇^g is the Levi-Civita connection for the metric g, φ is a section of the spinor bundle, and the dot denotes the Clifford action of tangent vectors on spinors. Spinors φ which are parallel with respect to $\widehat{\nabla}$ are called (imaginary) Killing spinors.

We will now describe the Killing spinors on hyperbolic space. The ball model of hyperbolic space is given by the metric $\omega^{-2}\xi$ where $\omega(x) = \frac{1}{2}(1 - |x|^2)$ and ξ is the flat metric on the open unit ball B^n in $\mathbb{R}^n$. In this model the Killing spinors on $\mathbb{H}^n$ are all spinors of the form

$$\varphi_s(x) = \omega(x)^{-1/2}(1 - ix\cdot)s.$$

Here s is a constant spinor on (B^n, ξ), or equivalently an element of the spinor representation space Σ. For the Clifford action we identify points in B^n with tangent vectors. For any Killing spinor φ_s on $\mathbb{H}^n$ its squared norm $V_s := |\varphi_s|^2$ is an element of $\mathcal{N}$. Every element of $\mathcal{N}$ of the form $V_{(0)} - \sum_{i=1}^n a_i V_{(i)}$ where $(a_1, \ldots, a_n) \in S^{n-1}$ is equal to V_s for some Killing spinor φ_s.

Using the connection $\widehat{\nabla}^g$ we define the Dirac operator $\widehat{D}^g$ by

$$\widehat{D}^g\varphi := \sum_{i=1}^n e_i \cdot \widehat{\nabla}^g_{e_i}\varphi = D^g\varphi - \frac{in}{2}\varphi,$$

where e_i, $i = 1, \ldots, n$, is an orthonormal frame for g and $D^g = \sum_{i=1}^n e_i \cdot \nabla^g_{e_i}$ is the Dirac operator associated to ∇^g. The Schrödinger-Lichnerowicz formula for $\widehat{D}^g$ has a boundary term related to the asymptotically hyperbolic mass. If (M, g) is an asymptotically hyperbolic manifold with diffeomorphism $\Phi : M \setminus K \to \mathbb{H}^n \setminus B$ at infinity, then the Killing spinor φ_s on $\mathbb{H}^n$ can be pulled back to a spinor $\Phi^*\varphi_s$ on $M \setminus K$. If ψ_s is a spinor on M with $\widehat{D}^g\psi_s = 0$ and $\psi_s - \Phi^*\varphi_s \to 0$ at infinity then the Schrödinger-Lichnerowicz formula for $\widehat{D}^g$ tells us that

$$\int_M \left(|\widehat{\nabla}^g\psi_s|^2 + \frac{\mathrm{Scal}^g + n(n-1)}{4}|\psi_s|^2\right) d\mu^g = \frac{1}{4}H_\Phi(V_s), \tag{21}$$

see [11, (4.11) and (4.22)].

We denote by H the space of positive smooth functions on $\mathbb{H}^n \setminus B_{R_0}$ which satisfy the Yamabe equation (3) and tend to 1 at infinity. In the proof of Theorem B we use the functionals $\mathcal{F}_s$ defined for $U \in H$ by

$$\mathcal{F}_s(U) := \inf\left\{\int_{\mathbb{H}^n \setminus B_{R_1}} |\widehat{\nabla}^g\psi|^2_g \, d\mu^g \; \middle| \; \psi - \Phi^*\varphi_s \to 0 \text{ at infinity}\right\}$$

where $g = U^\kappa b$ and $s \in \Sigma$. The infimum is attained by a spinor satisfying

$$\begin{cases} (\widehat{\nabla}^g)^*\widehat{\nabla}^g\psi := -\left(\nabla^g_{e_i} - \frac{i}{2}e_i\cdot\right)\left(\nabla^g_{e_i} + \frac{i}{2}e_i\cdot\right)\psi = 0, \\ \qquad \widehat{\nabla}^g_\nu\psi = 0 \text{ at the inner boundary of } (\mathbb{H}^n \setminus B_{R_1}, g), \\ \psi - \Phi^*\varphi_s \to 0 \text{ at infinity.} \end{cases}$$

The following Lemma is similar to [6, Lemma 12, page 231].

Lemma 4.2. $\mathcal{F}_s$ *is continuous with respect to the C^1 topology on H.*

Proof. Let U_1, U_2 be functions in H and set $g_1 = U_1^\kappa b$, $g_2 = U_2^\kappa b = W^\kappa g_1$, where $W := U_2/U_1$. Let ψ_1 and ψ_2 be the minimizers for $\mathcal{F}_s(U_1)$ and $\mathcal{F}_s(U_2)$. Using standard methods of identifying spinors for conformal metrics (see for example [19, Section 5.2]) we identify the spinor ψ_1 defined for the metric g_1 with the spinor

$\overline{\psi_1}$ for the metric g_2. Further, we can express the covariant derivative $\nabla^{g_2}\overline{\psi_1}$ as a leading term which is $\nabla^{g_1}\psi_1$ followed by terms involving dW and ψ_1. We then compute

$$
\begin{aligned}
\mathcal{F}_s(U_2) &= \int_{\mathbb{H}^n \setminus B_{R_1}} |\widehat{\nabla}^{g_2}\psi_2|^2_{g_2}\, d\mu^{g_2} \\
&\leq \int_{\mathbb{H}^n \setminus B_{R_1}} |\widehat{\nabla}^{g_2}\overline{\psi_1}|^2_{g_2}\, d\mu^{g_2} \\
&= \int_{\mathbb{H}^n \setminus B_{R_1}} |\widehat{\nabla}^{g_1}\psi_1|^2_{g_1}\, d\mu^{g_1} + E(W, \psi_1) \\
&= \mathcal{F}_s(U_1) + E(W, \psi_1).
\end{aligned}
$$

Here the remainder $E(W, \psi_1)$ is given by an integral over $\mathbb{H}^n \setminus B_{R_1}$ where each term in the integrand is quadratic in ψ_1 (containing ψ_1 or $\nabla^{g_1}\psi_1$) and contains one or two factors of the type $(1 - W^q)$ or dW. Since the minimizing spinor ψ for $\mathcal{F}_s(U)$ depends continuously on U we conclude that $E(W, \psi_1)$ can be made arbitrarily small by choosing U_2 sufficiently close to U_1 in C^1. By interchanging U_1 and U_2 we get an inequality in the other direction, and we conclude that $\mathcal{F}_s$ is continuous. $\qquad\square$

Let $s^\pm \in \Sigma$ be such that

$$
V_{s^\pm} = |\varphi_{s^\pm}|^2 = V_{(0)} \pm V_{(1)}.
$$

We define the functional $\mathcal{F}$ by

$$
\mathcal{F}(U) := \mathcal{F}_{s^+}(U) + \mathcal{F}_{s^-}(U).
$$

In the next Lemma we prove that the mass bounds $\mathcal{F}(U)$.

Lemma 4.3. *For $(M, g, \Phi, U) \in \mathcal{A}^{\mathrm{Spin}}$ we have*

$$
\mathcal{F}(U) \leq (n-1)\omega_{n-1}m^g.
$$

Proof. Since the integral in the definition of $\mathcal{F}_s(U)$ lacks the non-negative term involving scalar curvature and is taken over a smaller domain it is never larger than the integral in (21). Further, the infimum in the definition of $\mathcal{F}_s(U)$ can only decrease the value of the integral in (21) and we conclude that

$$
\mathcal{F}_{s^\pm}(U) \leq \frac{1}{4}H_\Phi(V_{s^\pm}).
$$

Therefore

$$
\begin{aligned}
\mathcal{F}(U) &= \mathcal{F}_{s^+}(U) + \mathcal{F}_{s^-}(U) \\
&\leq \frac{1}{4}\left(H_\Phi(V_{s^+}) + H_\Phi(V_{s^-})\right) \\
&= \frac{1}{4}\left(H_\Phi(V_{(0)} + V_{(1)}) + H_\Phi(V_{(0)} - V_{(1)})\right) \\
&= \frac{1}{2}H_\Phi(V_{(0)}) \\
&= (n-1)\omega_{n-1}m^g
\end{aligned}
$$

follows from (2). $\qquad\square$

Lemma 4.4. $\mathcal{F}(U) = 0$ *if and only if* $U \equiv 1$.

Proof. If $\mathcal{F}(U) = 0$ then $\mathcal{F}_{s^+}(U) = 0$ and $\mathcal{F}_{s^-}(U) = 0$ and both the infima are attained by non-trivial Killing spinors. The existence of a non-trivial Killing spinor implies that g is an Einstein metric with scalar curvature $-n(n-1)$. Since the metric g is also conformally flat it must have constant negative curvature -1. Since $U \to 1$ at infinity we conclude that $U \equiv 1$. If $U \equiv 1$ then g is the hyperbolic metric which has Killing spinors, and thus $\mathcal{F}(U) = 0$. $\qquad\square$

Proof of Theorem B. As in the proof of Theorem A we argue by contradiction. We assume that there is a sequence (M_k, g_k, Φ_k, U_k) of elements of $\mathcal{A}^{\mathrm{Spin}}$ such that m^{g_k} tends to zero while $|U_k - 1| \geq \varepsilon$. Arguing as in the proof of Theorem A, a subsequence of $U_k - 1$ will converge to a limit $U_\infty - 1$ in $X_\delta^{2,p}(\mathbb{H}^n \setminus B_{R_1})$ for which $\sup_{\mathbb{H}^n \setminus B_{R_1}} |U_\infty - 1| \geq \varepsilon$. From Lemma 4.2 we see that $\lim_{k\to\infty} \mathcal{F}(U_k) = \mathcal{F}(U_\infty)$, and from Lemma 4.3 we have $\lim_{k\to\infty} \mathcal{F}(U_k) = 0$. Lemma 4.4 then tells us that $U_\infty \equiv 1$ which is a contradiction. From this we conclude that for every $\varepsilon > 0$ there is a $\delta > 0$ such that for (M, g, Φ, U) belonging to $\mathcal{A}^{\mathrm{Spin}}$ with $m^g \leq \delta$ it holds that $\sup_{\mathbb{H}^n \setminus B_{R_1}} |U - 1| \leq \varepsilon$. The Theorem now follows from Proposition 3.4. $\qquad\square$

APPENDIX A. THE ANTI-DE SITTER-SCHWARZSCHILD SPACETIME

In this appendix, we discuss the Anti-de Sitter-Schwarzschild metrics in dimension n following [24, Section 2] where the case $n = 3$ is treated. These metrics are also called Kottler metrics with negative cosmological constant. Furthermore we explicit the lapse function, see [25] for the $3 + 1$-dimensional case.

A.1. **The metric in areal coordinate.** Let g be a Riemannian metric on the n-dimensional manifold M and let

$$\gamma := -V^2 dt^2 + g$$

be a Lorentzian metric defined on manifold $\mathcal{M} := \mathbb{R} \times M$. If we assume that the function V does not depend on t then γ solves the Einstein equations with cosmological constant Λ,

$$\mathrm{Ric}^\gamma - \frac{\mathrm{Scal}^\gamma}{2}\gamma + \Lambda\gamma = 0,$$

if and only if

$$\mathrm{Scal}^g = 2\Lambda, \tag{22a}$$

$$\mathrm{Ric}^g - \frac{\mathrm{Hess}^g V}{V} + \frac{\Delta^g V}{V}g = 0. \tag{22b}$$

Such a metric g is called static. See also [11, Equations (0.1)-(0.2)]. In what follows we will always assume that when the metric is not indicated, the curvature tensors and the connection are defined with respect to the metric g. We are interested in the case of negative cosmological constant. We assume that

$$\Lambda = -\frac{n(n-1)}{2},$$

which can always be achieved by a rescaling. From Equation (22a), this imposes $\mathrm{Scal}^g = -n(n-1)$. Taking the trace of Equation (22b) yields $\Delta V = nV$ so we can

rewrite the system as

$$\mathrm{Scal}^g = -n(n-1), \tag{23a}$$

$$\mathrm{Ric}^g_{ij} - \frac{\mathrm{Hess}^g_{ij} V}{V} + n g_{ij} = 0. \tag{23b}$$

Note that the slice $t = 0$ is totally geodesic. In particular, marginally (outer) trapped surfaces correspond to minimal surfaces for the metric g. We now assume that the metric g is rotationally symmetric. Such a metric can be written in full generality as $g = ds^2 + k(s)^2 \sigma$ where σ is the round metric on S^{n-1}. The mean curvature of a surface of constant s is given by $H(s) = (n-1)\frac{\partial_s k}{k}$. So, in a region where no surface of constant s is a minimal surface, k has non-vanishing derivative and we can use it as a radial (areal) coordinate ρ so that

$$g = f(\rho)^2 d\rho^2 + \rho^2 \sigma.$$

In what follows we assume that the coordinate index 1 corresponds to the ρ-coordinate while upper-case latin letters represent coordinates on the sphere and run from 2 to n. The Christoffel symbols of the metric g are given by

$$\begin{cases} \Gamma^1_{11} = \dfrac{f'}{f}, \\[2mm] \Gamma^1_{1A} = 0, \\[2mm] \Gamma^A_{11} = 0, \\[2mm] \Gamma^A_{1B} = \dfrac{1}{\rho}\delta^A_B, \\[2mm] \Gamma^1_{AB} = -\dfrac{\rho}{f^2}\sigma_{AB}, \\[2mm] \Gamma^C_{AB} = \gamma^C_{AB}, \end{cases}$$

where γ^C_{AB} are the Christoffel symbols of the metric σ. The components of the curvature tensors of the metric g can then be computed,

$$\begin{cases} \mathcal{R}^L{}_{KIJ} = \left(1 - \dfrac{1}{f^2}\right)\left(\delta^L_I \sigma_{JK} - \delta^L_J \sigma_{IK}\right), \\[3mm] \mathcal{R}^0{}_{KIJ} = 0, \\[3mm] \mathcal{R}^0{}_{K0J} = \dfrac{\rho f'}{f^3}\sigma_{JK}, \\[3mm] \mathrm{Ric}_{KJ} = \left[(n-2)\left(1 - \dfrac{1}{f^2}\right) + \dfrac{\rho f'}{f^3}\right]\sigma_{JK}, \\[3mm] \mathrm{Ric}_{1J} = 0, \\[3mm] \mathrm{Ric}_{11} = (n-1)\dfrac{\rho f'}{f}, \\[3mm] \mathrm{Scal} = 2(n-1)\dfrac{f'}{\rho f^3} + \dfrac{(n-1)(n-2)}{\rho^2}\left(1 - \dfrac{1}{f^2}\right). \end{cases}$$

What is interesting about these formulas is that the curvature tensor depends only on the first derivative of f. In particular, Equation (23b) becomes

$$-n = 2\frac{f'}{\rho f^3} + \frac{n-2}{\rho^2}\left(1 - \frac{1}{f^2}\right). \tag{24}$$

Defining u through $f(\rho) = u(\rho)^{-\frac{1}{2}}$ we get

$$\frac{u'}{\rho} + (n-2)\frac{u-1}{\rho^2} = n.$$

The general solution of this equation is given by

$$u(\rho) = 1 + \rho^2 - \frac{2m}{\rho^{n-2}}$$

where m is a free parameter that can be identified with the mass. Our next goal is to find the lapse function $V = V(\rho)$. Equation (23b) can be decomposed into radial, tangential, and mixed terms. The mixed terms vanish while the other two lead to the following equations.

$$\begin{cases} 0 = (n-1)V\dfrac{\partial_1 f}{\rho f} - \partial_\rho^2 V + \dfrac{\partial_\rho f}{f}\partial_\rho V + nf^2 V, \\[3mm] 0 = V\left[(n-2)\left(1 - \dfrac{1}{f^2}\right) + \dfrac{\rho \partial_1 f}{f^3}\right] - \dfrac{\rho}{f^2}\partial_1 V + n\rho^2 V. \end{cases} \qquad (25)$$

The second equation can be combined with Equation (24) to yield

$$\frac{V'}{V} = -\frac{f'}{f}.$$

Hence $V = \frac{\lambda}{f}$. Up to a redefinition of t we can assume that $V = \frac{1}{f}$. It is then a simple calculation to see that the first line of Equation (25) is also fulfilled. Hence the Anti-de Sitter-Schwarzschild metric can be written as follows,

$$\gamma_{\mathrm{AdSS}} = -\left(1 + \rho^2 - \frac{2m}{\rho^{n-2}}\right)dt^2 + \frac{d\rho^2}{1 + \rho^2 - \frac{2m}{\rho^{n-2}}} + \rho^2\sigma.$$

We now study separately the cases $m > 0$ and $m < 0$.

A.2. **The case of positive mass.** The metric g is only defined on the set $\{\rho \geq a(m)\}$ where $a(m)$ is the unique solution of

$$1 + \rho^2 - \frac{2m}{\rho^{n-2}} = 0.$$

We define

$$h_m(\rho) = \int_\rho^\infty \frac{ds}{s\sqrt{1 + s^2 - \frac{2m}{s^{n-2}}}}$$

and the functions r and φ by

$$e^r = \frac{1 + e^{-h_m(\rho)}}{1 - e^{-h_m(\rho)}}, \qquad \varphi^{\frac{2}{n-2}} = \frac{\rho}{\sinh(r)}.$$

We note that $r \to r_0(m) > 0$ when $\rho \to a(m)^+$. The function $r : (a(m), \infty) \to (r_0(m), \infty)$ is a smooth increasing function of ρ. We remark that

$$\begin{cases} \rho = \varphi^{\frac{2}{n-2}}\sinh(r), \\[3mm] \dfrac{d\rho}{\rho\sqrt{1 + \rho^2 - \frac{2m}{\rho^{n-2}}}} = \dfrac{dr}{\sinh(r)}. \end{cases} \qquad (26)$$

The metric g can then be written

$$g = \varphi^{\frac{4}{n-2}}\left(dr^2 + \sinh(r)^2\sigma\right).$$

The mean curvature of the hypersurfaces of constant ρ is given by

$$H = \varphi^{-\frac{2}{n-2}} \left[(n-1) \coth(r) + \frac{2(n-1)}{n-2} \frac{\partial_r \varphi}{\varphi} \right]. \tag{27}$$

A simple calculation shows that φ and $\partial_r \varphi$ are continuous at $r = r_0(m)$ and that the hypersurface $r = r_0(m)$ is a minimal surface.

We next show that the manifold can be doubled to a complete asymptotically hyperbolic manifold of constant scalar curvature. For this we first switch to the conformal ball model of hyperbolic space and set

$$\tau = \frac{e^r - 1}{e^r + 1}.$$

The metric g becomes

$$g = \frac{4\varphi^{\frac{4}{n-2}}}{(1-\tau^2)^2} \left(d\tau^2 + \tau^2 \sigma \right)$$

and is defined on the annulus $b(m) \leq |x| < 1$ in $\mathbb{R}^n$ where $\tau = |x|$ and $b(m)$ is given by

$$b(m) = \frac{e^{r_0(m)} - 1}{e^{r_0(m)} + 1}.$$

As is well known, the inversion

$$\begin{aligned} i: \quad \mathbb{R}^n \setminus \{0\} \quad &\to \quad \mathbb{R}^n \setminus \{0\} \\ x \quad &\mapsto \quad \frac{b(M)^2}{|x|^2} x \end{aligned}$$

is a conformal transformation. Pulling back the metric g to the annulus $b^2(m) < |x| \leq b(m)$ by the inversion, we get the following extension of the metric g,

$$g = \frac{4 f_m^{\frac{4}{n-2}}}{(1-\tau^2)^2} \left(d\tau^2 + \tau^2 \sigma \right),$$

where

$$f_m^{\frac{2}{n-2}}(\tau) = \begin{cases} \varphi^{\frac{2}{n-2}}(\tau) & \text{if } b(m) \leq \tau \leq 1, \\ b(m)^2 \varphi^{\frac{2}{n-2}} \left(\frac{b(m)^2}{\tau} \right) \frac{1-\tau^2}{\tau^2 - b(m)^4} & \text{if } b(m)^2 \leq \tau \leq b(m). \end{cases}$$

In the following propositions we collect the basic properties of the metric g_{AdSS}. Most of them are useful in the course of the proof of our main results. See also [24, Section 2] in for the three-dimensional case.

Proposition A.1. *For each $m > 0$ the Anti-de Sitter-Schwarzschild metric is asymptotically hyperbolic and is defined on $\mathbb{H}^n \setminus B_{b(m)^2}$. Moreover,*

1. *$f_m > 1$, $\lim_{\tau \to 1} f_m(\tau) = 1$, $\lim_{\tau \to b(m)^2} f_m(\tau) = \infty$.*
2. *There exists a constant $C > 0$ independent of m such that $f_m \leq 1 + Cme^{-nr}$ provided that $r \geq r_1(m)$, where r_1 is a nondecreasing continuous function of m such that $r_1(m) > r_0(m)$. Consequently, $f_m = 1 + O(e^{-nr})$ when $r \to \infty$.*
3. *g_{AdSS} has constant scalar curvature $-n(n-1)$ and mass m.*
4. *$\partial_r f_m < 0$.*
5. *The hypersurface $r = r_0(m)$ is the only compact minimal surface.*

Proof. The mass of g_{AdSS} is easily computed using [11, Formula (2.25)], and Property 3 follows.

Fixing $r \geq r_0(m)$, we remark that for all $s \in (a(m), \infty)$,

$$h_m(s) > h_0(s) = \frac{1}{2} \ln \frac{\sqrt{1+s^2}+1}{\sqrt{1+s^2}-1}.$$

In particular, if $\sinh(r) \geq a(m)$ we get

$$h_m(\sinh(r)) > \ln\left(\coth\frac{r}{2}\right).$$

Since $h_m(\rho) = \ln \coth \frac{r}{2}$, and since h_m is decreasing, we have $\rho > \sinh(r)$. Obviously $\sinh(r) < \rho$ if $\sinh(r) < a(m)$. This proves that $\varphi = \frac{\rho}{\sinh(r)} > 1$ for any $r \geq r_0(m)$.

We next find an upper bound for f_m. First, it is clear that $a(m) < (2m)^{1/n}$. If we assume that $\rho \geq (2m)^{1/n}$ then $\rho > a(m)$ and

$$\begin{aligned}
h_m(\rho) &= \int_\rho^\infty \frac{ds}{s\sqrt{1+s^2}\sqrt{1-\frac{2m}{(1+s^2)s^{n-2}}}} \\
&\leq \frac{1}{\sqrt{1-m\rho^{-n}}} \int_\rho^\infty \frac{ds}{s\sqrt{1+s^2}} \\
&= \frac{\operatorname{arcsinh}(\rho^{-1})}{\sqrt{1-m\rho^{-n}}}.
\end{aligned}$$

Observe also that $\sinh r = \frac{1}{\sinh h_m(\rho)}$, hence $\frac{\rho}{\sinh r} = \rho \sinh h_m(\rho)$. Set

$$\eta(t) := \sinh\left(\frac{\operatorname{arcsinh}(\rho^{-1})}{\sqrt{1-t}}\right).$$

Let $R_1 > 0$ be fixed, and assume that $\rho \geq \max\{r^{-1}(R_1), (2m)^{1/n}\}$. Using the mean value theorem and the inequality $\operatorname{arcsinh}(\rho^{-1}) \leq \rho^{-1}$ we have

$$\begin{aligned}
\sinh h_m(\rho) &= \eta(m\rho^{-n}) \\
&\leq \eta(0) + m\rho^{-n} \sup_{0\leq\theta\leq 1} \eta'(\theta m\rho^{-n}) \\
&= \rho^{-1} + m\rho^{-n} \sup_{0\leq\theta\leq 1}\left(\frac{\operatorname{arcsinh}(\rho^{-1})}{2(1-\theta m\rho^{-n})^{3/2}} \cosh\left(\frac{\operatorname{arcsinh}(\rho^{-1})}{\sqrt{1-\theta m\rho^{-n}}}\right)\right) \\
&\leq \rho^{-1}\left(1 + Cm\rho^{-n}\right),
\end{aligned}$$

for some constant $C > 0$ which does not depend on m. Since $\rho \geq \sinh r$ it is now easy to check that $f_m = (\rho \sinh h_m(\rho))^{\frac{n-2}{2}} \leq 1 + Cme^{-nr}$ (possibly for a larger constant $C > 0$), provided that $r \geq r_1(m) := \max\{R_1, r((2m)^{1/n})\}$. By definition, it is clear that $r_1(m) > r_0(m)$. The second statement is thereby proved.

Next, f_m solves the Yamabe equation

$$-\frac{4(n-1)}{n-2}\Delta^b f_m + n(n-1)\left(f_m^{\frac{n+2}{n-2}} - f_m\right) = 0.$$

In polar normal coordinates we have $\sqrt{\det(b)} = \sinh(r)^{n-1}$. Hence, from the well known formula

$$\Delta f_m = \frac{1}{\sqrt{\det(b)}}\partial_i\left(\sqrt{\det(b)}b^{ij}\partial_j f_m\right)$$

we infer that

$$\partial_r \left(\sinh(r)^{n-1} \partial_r f_m \right) = \frac{n(n-2)}{4} \sinh(r)^{n-1} \left(f_m^{\frac{n+2}{n-2}} - f_m \right).$$

Assume that $\partial_r f_m(\bar{r}) \geq 0$ for some $\bar{r}$. Then, $\partial_r f_m(r) > 0$ for all $r > \bar{r}$ since $f_m > 1$. This contradicts the fact that $f_m \to 1$ when $r \to \infty$, $f_m > 1$. Hence, $\partial_r f_m < 0$ for all r.

We finally prove that the hypersurface $r = r_0(m)$ is the only minimal surface. From Formula (27), the sphere of constant ρ has mean curvature

$$H(\rho) = (n-1)\sqrt{1 + \frac{1}{\rho^2} - \frac{2m}{\rho^n}}.$$

For any $\rho > a(m)$ we have $H(\rho) > 0$. Thus by the maximum principle for minimal surfaces, if Σ is a minimal surface, then $\sup_\Sigma \rho \leq 0$, that is $\sup_\Sigma \tau \leq b(m)$. By symmetry, we also have that $\inf_\Sigma \tau \leq b(m)$. This proves that Σ coincides with the sphere $r = r_0(m)$. $\square$

Proposition A.2. $a(m)$, $r_0(m)$ and $b(m)$ are continuous increasing functions of m. Further,

1. $a(m), r_0(m), b(m) \to 0$ as $m \to 0$,
2. $a(m), r_0(m) \to \infty$ and $b(m) \to 1$ as $m \to \infty$.

Proof. It is easy to see that $a(m)$ is a continuous increasing function of m. Since the function $\rho \mapsto 1 + \rho^2 - \frac{2m}{\rho^{n-2}}$ is increasing, we know that $\rho_-(m) \leq a(m) \leq \rho_+(m)$ provided that

$$\begin{cases} 0 \leq 1 + \rho_+^2 - \dfrac{2m}{\rho_+^{n-2}}, \\[2mm] 0 \geq 1 + \rho_-^2 - \dfrac{2m}{\rho_-^{n-2}}. \end{cases}$$

One can select $\rho_+ = m^{1/n}$. Assuming $m > 1$, we choose

$$\rho_- = (2m)^{1/n}\sqrt{1 - \frac{1}{(2m)^{2/n}}}.$$

Simple computations shows that both inequalities are fulfilled. Hence for large m, $a(m) \sim (2m)^{1/n}$. For small positive m, we obviously have $0 < a(m) < \rho_+(m)$. So $a(m) \to 0$ when $m \to 0^+$.

We next turn our attention to the function r_0. We first give an upper bound for $h_m(a(m))$ as follows. Remark that on the interval $(a(m), \infty)$ we have

$$1 + s^2 - \frac{2m}{s^{n-2}} \geq 1 + s^2 - \frac{2m}{a(m)^{n-2}}$$
$$= 1 + s^2 - (1 + a(m)^2)$$
$$= s^2 - a(m)^2.$$

Hence,

$$\ln\left(\coth \frac{r_0(m)}{2} \right) = h_m(a(m)) \leq \int_{a(m)}^{\infty} \frac{ds}{s\sqrt{s^2 - a(m)^2}} = \frac{\pi}{2a(m)}.$$

This implies that $r_0(m) \to \infty$ as $m \to \infty$.

In order to estimate r_0 when $m \to 0^+$ we give a lower bound for $h_m(a(m))$,

$$
\ln\left(\coth\frac{r_0(m)}{2}\right) = h_m(a(m))
$$

$$
= \int_{a(m)}^{\infty} \frac{ds}{s\sqrt{1 + s^2 - \frac{a(m)^n + a(m)^{n-2}}{s^{n-2}}}}
$$

$$
= \int_{a(m)}^{\infty} \frac{ds}{s^{2-\frac{n}{2}}\sqrt{s^n - a(m)^n + s^{n-2} - a(m)^{n-2}}}
$$

$$
\geq \int_{a(m)}^{\infty} \frac{ds}{s^{2-\frac{n}{2}}\sqrt{(s - a(m))(ns^{n-1} + (n-2)s^{n-3})}}
$$

$$
\geq \int_{a(m)}^{\infty} \frac{ds}{\sqrt{s}\sqrt{(s - a(m))(ns^2 + (n-2))}}
$$

$$
\geq \frac{1}{\sqrt{2n-2}} \int_{a(m)}^{1} \frac{ds}{\sqrt{s(s - a(m))}}
$$

$$
\geq \frac{1}{\sqrt{2n-2}} \int_{1}^{\frac{1}{a(m)}} \frac{dt}{\sqrt{t(t-1)}}.
$$

It is obvious that the last integral diverges when $a(m) \to 0^+$. Hence $r_0(m) \to 0$ when $m \to 0$.

The limits of b follow from the relation $b(m) = \frac{e^{r_0(m)} - 1}{e^{r_0(m)} + 1}$. $\qquad\square$

A.3. **The case of negative mass.** Remark that when $m < 0$ the function $h(m)$ tends to a finite positive value at $r = 0$. Changing to the ρ coordinate, this means that the metric $g = \varphi^{\frac{4}{n-2}}(dr^2 + \sinh(r)^2\sigma)$ is only defined for $r \geq r_0(m)$ such that

$$
h_m(0) = \int_0^{\infty} \frac{ds}{s\sqrt{1 + s^2 - \frac{2m}{r^{n-2}}}} = \ln\frac{1 + e^{-r_0}}{1 - e^{-r_0}}.
$$

The function φ satisfies $\varphi(r_0(m)) = 0$.

Proposition A.3. *The function $m \mapsto r_0(m)$ is continuous and strictly decreasing on the interval $(-\infty, 0)$. Further,*

1.
$$
\begin{cases}
\lim_{m\to 0^-} r_0(m) = 0 \\
\lim_{m\to -\infty} r_0(m) = \infty
\end{cases}
$$

2. *The function $f_m := \varphi : \mathbb{H}^n \setminus B_{r_0(m)}(0) \to \mathbb{R}_+$ solves the Yamabe equation with zero boundary value on $\partial B_{r_0(m)}(0)$ and satisfies $f_m < 1$.*
3. *There exists a constant $C > 0$ independent of m such that $f_m \geq 1 - Cme^{-nr}$ provided that $r \geq r_1(m)$, where r_1 is a nonincreasing continuous function of m such that $r_1(m) > r_0(m)$. Consequently, $f_m = 1 + O(e^{-nr})$ when $r \to \infty$.*
4. $\partial_r f_m > 0$.

Proof. We remark that the integrand is positive and strictly increasing with respect to m. From dominated convergence, it is easy to argue that $m \mapsto h_m(0)$ is continuous. When $|m| \to \infty$, the integrand tends to 0 so $\lim_{m\to-\infty} h_m(0) = 0$. This forces

$\lim_{m \to -\infty} r_0(m) = \infty$. Similarly, when $m \to 0^-$, by the monotone convergence theorem,

$$h_m(0) \to \int_0^\infty \frac{ds}{s\sqrt{1+s^2}} = \infty.$$

Hence, $\lim_{m \to 0^-} r_0(m) = 0$.

The properties of f_m follow in the same manner as their counterparts in the case $m > 0$ (see Proposition A.1). We only remark (for a future reference) that having fixed $R_1 > 0$ one may define $r_1(m)$ as $r_1(m) := \max\{R_1, r((-Cm)^{1/n})\}$, where the constant $C > 0$ depends on R_1 only. It is then obvious that $r_1(m) > r_0(m)$.   □

A.4. **A characterization of Anti-de Sitter-Schwarzschild spacetimes.** In this section, we give a characterization of Anti-de Sitter-Schwarzschild metrics which is useful in the proof of Theorem A. See [20] for similar results.

Proposition A.4. *Let K be a compact subset of $\mathbb{H}^n$ such that $\mathbb{H}^n \setminus K$ is connected and U, V two functions defined on $\mathbb{H}^n \setminus K$. Let $g := U^\kappa b$. Assume that the metric*

$$-V^2 dt^2 + g$$

is static with cosmological constant

$$\Lambda = -\frac{n(n-1)}{2}.$$

Assume further that the function U is bounded from above and away from zero and that the function V is positive, tends to infinity at infinity and has no critical point outside a compact set. Then there is a point $x_0 \in \mathbb{H}^n$ and $m \in \mathbb{R}$ such that

$$U = f_m(r),$$

where $r := d^b(x_0, \cdot)$.

Proof. In what follows, covariant derivatives and curvatures are defined with respect to the metric g unless stated otherwise.

Since $g = U^\kappa b$ on $\mathbb{H}^n \setminus B_{R_1}$ is conformally flat, it has vanishing Cotton-York tensor (see for example [9, Proposition 1.62]). Since g has constant scalar curvature this is equivalent to

$$\nabla_i \widehat{\mathrm{Ric}}_{jk} - \nabla_j \widehat{\mathrm{Ric}}_{ik} = 0. \tag{28}$$

From the static equations (23a)-(23b) it follows that

$$\widehat{\mathrm{Ric}} = \frac{\overset{\circ}{\mathrm{Hess}}(V)}{V},$$

where $\widehat{\mathrm{Ric}} := \mathrm{Ric} + (n-1)g$ denotes the traceless Ricci tensor and $\overset{\circ}{\mathrm{Hess}}(V)$ denotes the traceless Hessian of V (which in index notation is denoted also by $\overset{\circ}{\nabla}_{i,j} V$). From (28) and the fact that

$$\frac{\mathrm{Hess}(V)}{V} = \frac{1}{V}\left(\overset{\circ}{\mathrm{Hess}} V + \frac{\Delta V}{n} g\right) = \frac{\overset{\circ}{\mathrm{Hess}}(V)}{V} + g,$$

we conclude

$$
\begin{aligned}
0 &= \nabla_i \frac{\mathring{\nabla}_{j,k} V}{V} - \nabla_j \frac{\mathring{\nabla}_{i,k} V}{V} \\
&= \nabla_i \frac{\nabla^2_{j,k} V}{V} - \nabla_j \frac{\nabla^2_{i,k} V}{V} \\
&= \frac{\nabla_i \nabla_j \nabla_k V}{V} - \frac{\nabla_j \nabla_i \nabla_k V}{V} - \frac{\nabla^2_{j,k} V}{V} \frac{\nabla_i V}{V} + \frac{\nabla^2_{i,k} V}{V} \frac{\nabla_j V}{V} \\
&= -\mathcal{R}^l{}_{kij} \frac{\nabla_l V}{V} - \frac{\nabla^2_{j,k} V}{V} \frac{\nabla_i V}{V} + \frac{\nabla^2_{i,k} V}{V} \frac{\nabla_j V}{V}.
\end{aligned}
$$

Since g is conformally flat its Weyl tensor vanishes, so

$$
\mathcal{R} = \frac{\mathrm{Scal}}{2n(n-1)} g \owedge g + \frac{1}{n-2} \widehat{\mathrm{Ric}} \owedge g = -\frac{1}{2} g \owedge g + \frac{1}{n-2} \widehat{\mathrm{Ric}} \owedge g,
$$

where $\owedge$ denotes the Kulkarni-Nomizu product (see for example [5, Definition 1.110]). As a consequence, we get

$$
\begin{aligned}
0 &= -\mathcal{R}^l{}_{kij} \frac{\nabla_l V}{V} - \frac{\nabla^2_{j,k} V}{V} \frac{\nabla_i V}{V} + \frac{\nabla^2_{i,k} V}{V} \frac{\nabla_j V}{V} \\
&= (g_{li} g_{kj} - g_{lj} g_{ki}) \frac{\nabla^l V}{V} - \frac{\nabla^2_{j,k} V}{V} \frac{\nabla_i V}{V} + \frac{\nabla^2_{i,k} V}{V} \frac{\nabla_j V}{V} \\
&\quad - \frac{1}{n-2} \left(\widehat{\mathrm{Ric}}_{li} g_{kj} + \widehat{\mathrm{Ric}}_{kj} g_{li} - \widehat{\mathrm{Ric}}_{lj} g_{ki} - \widehat{\mathrm{Ric}}_{ki} g_{lj} \right) \frac{\nabla^l V}{V} \\
&= -\frac{\mathring{\nabla}_{j,k} V}{V} \frac{\nabla_i V}{V} + \frac{\mathring{\nabla}_{i,k} V}{V} \frac{\nabla_j V}{V} \\
&\quad - \frac{1}{n-2} \left(\frac{\mathring{\nabla}_{l,i} V}{V} g_{kj} + \frac{\mathring{\nabla}_{k,j} V}{V} g_{li} - \frac{\mathring{\nabla}_{l,j} V}{V} g_{ki} - \frac{\mathring{\nabla}_{k,i} V}{V} g_{lj} \right) \frac{\nabla^l V}{V} \\
&= \frac{n-1}{n-2} \left(\frac{\mathring{\nabla}_{i,k} V}{V} \frac{\nabla_j V}{V} - \frac{\mathring{\nabla}_{j,k} V}{V} \frac{\nabla_i V}{V} \right) \\
&\quad - \frac{1}{n-2} \left(\frac{\mathring{\nabla}_{l,i} V}{V} \frac{\nabla^l V}{V} g_{kj} - \frac{\mathring{\nabla}_{l,j} V}{V} \frac{\nabla^l V}{V} g_{ki} \right).
\end{aligned}
\tag{29}
$$

We set $\xi_i := \frac{\mathring{\nabla}_{i,j} V}{V} \frac{\nabla^j V}{V}$. Contracting the previous equation with $\frac{\nabla^k V}{V}$ we get

$$
0 = \xi_i \frac{\nabla_j V}{V} - \xi_j \frac{\nabla_i V}{V}.
$$

This is possible only if ξ and $\frac{\nabla V}{V}$ are colinear. We let λ be the function such that $\xi = (n-1)\lambda \frac{\nabla V}{V}$. Equation (29) then implies

$$
0 = \left(\frac{\mathring{\nabla}_{i,k} V}{V} + \lambda g_{ik} \right) \frac{\nabla_j V}{V} - \left(\frac{\mathring{\nabla}_{j,k} V}{V} + \lambda g_{jk} \right) \frac{\nabla_i V}{V}.
$$

This is possible only if

$$
\frac{\mathring{\nabla}_{i,k} V}{V} + \lambda g_{ik} = \mu \frac{\nabla_i V}{V} \frac{\nabla_j V}{V}
$$

for some function μ. The trace of this last equation gives a direct relation between λ and μ,

$$\lambda = \frac{\mu}{n}\left|\frac{dV}{V}\right|^2.$$

Hence,

$$\widehat{\mathrm{Ric}}_{ij} = \frac{\overset{\circ}{\nabla}_{i,j}V}{V} = \mu\frac{\nabla_i V}{V}\frac{\nabla_j V}{V} - \frac{\mu}{n}\left|\frac{dV}{V}\right|^2 g_{ij}.$$

By a straightforward calculation, we have

$$\nabla_i\left|\frac{dV}{V}\right|^2 = 2\left(1+\left[\mu\left(1-\frac{1}{n}\right)-1\right]\left|\frac{dV}{V}\right|^2\right)\frac{\nabla_i V}{V}. \tag{30}$$

We now choose V_0 to be such that V has no critical point outside $V^{-1}(-\infty, V_0)$. We remark that $V^{-1}(V_0, \infty)$ is connected. Indeed if it was not, from the assumption that V is proper it would have one bounded connected component Ω. Since $V = V_0$ on $\partial\Omega$, V reaches a local maximum on Ω which contradicts the assumption that V has no critical point on Ω. We let Σ_0 be the boundary of a connected component of $V^{-1}(-\infty, V_0)$. Equation (30) shows that $\left|\frac{dV}{V}\right|^2$ is constant along Σ_0. Plugging Equation (30) into (29), we get

$$0 = \nabla_i\mu\left(\frac{\nabla_j V}{V}\frac{\nabla_k V}{V} - \frac{1}{n}\left|\frac{dV}{V}\right|^2 g_{jk}\right) - \nabla_j\mu\left(\frac{\nabla_i V}{V}\frac{\nabla_k V}{V} - \frac{1}{n}\left|\frac{dV}{V}\right|^2 g_{ik}\right)$$
$$+ \mu\underbrace{\left[1+\frac{2}{n}\left(1-\left|\frac{dV}{V}\right|^2\right)+\frac{\mu}{n}\left(1-\frac{2}{n}\right)\left|\frac{dV}{V}\right|^2\right]}_{=:\,\theta}\left(g_{ik}\frac{\nabla_j V}{V} - g_{jk}\frac{\nabla_i V}{V}\right).$$

Taking the trace of this last equation with respect to j and k we get

$$\frac{1}{n}\left|\frac{dV}{V}\right|^2\nabla_i\mu = \left[\left\langle d\mu, \frac{dV}{V}\right\rangle + (n-1)\theta\mu\right]\frac{\nabla_i V}{V}.$$

This implies that μ is constant on the hypersurface Σ_0. The second fundamental form S of Σ_0 is equal to the normalized Hessian of V restricted to $T\Sigma_0$, that is

$$S_{ij} = \frac{\nabla^2_{i,j}V}{|dV|} = \frac{V}{|dV|}\left(1-\frac{\mu}{n}\left|\frac{dV}{V}\right|^2\right)g_{ij}. \tag{31}$$

Hence the hypersurface Σ_0 is umbilic with constant mean curvature. From the conformal transformation law of the second fundamental form, Σ_0 is umbilic for the hyperbolic metric b as well. Since it is also compact it is a round sphere.

Note that the curvature of Σ_0 is given by the Gauss Formula,

$$\mathcal{R}^{\Sigma_0} = \mathcal{R} + \frac{1}{2}S \oslash S.$$

From the form of the Riemann tensor of g and the special form of S, we immediately conclude that the metric induced on Σ_0 has constant curvature,

$$
\begin{aligned}
\mathcal{R}^{\Sigma_0} &= -\frac{1}{2}g \owedge g + \frac{1}{n-2}\widehat{\mathrm{Ric}} \owedge g + \frac{1}{2}\frac{\mathrm{Hess}(V)}{|dV|} \owedge \frac{\mathrm{Hess}(V)}{|dV|} \\
&= -\frac{1}{2}g \owedge g + \frac{1}{n-2}\frac{\mathring{\mathrm{Hess}}(V)}{V} \owedge g + \frac{1}{2}\frac{V^2}{|dV|^2}\left(\frac{\mathring{\mathrm{Hess}}(V)}{V} + g\right) \owedge \left(\frac{\mathring{\mathrm{Hess}}(V)}{V} + g\right) \\
&= -\frac{1}{2}g \owedge g - \frac{1}{n-2}\frac{\mu}{n}\left|\frac{dV}{V}\right|^2 g \owedge g + \frac{1}{2}\frac{V^2}{|dV|^2}\left(1 - \frac{\mu}{n}\left|\frac{dV}{V}\right|^2\right)^2 g \owedge g \\
&= \left[-\frac{1}{2} - \frac{1}{n-2}\frac{\mu}{n}\left|\frac{dV}{V}\right|^2 + \frac{1}{2}\frac{V^2}{|dV|^2}\left(1 - \frac{\mu}{n}\left|\frac{dV}{V}\right|^2\right)^2\right] g \owedge g,
\end{aligned}
$$

where we used the fact that

$$
\frac{\mathring{\mathrm{Hess}}(V)}{V} = -\frac{\mu}{n}\left|\frac{dV}{V}\right|^2 g
$$

when restricted to $T\Sigma_0$.

We claim that the level set $V^{-1}(V_0)$ is connected. Assume that it contains at least two connected components Σ_0, Σ_1. Since $V^{-1}(V_0, \infty)$ is connected and open, it is path connected so we can join Σ_0 and Σ_1 by a path γ in $V^{-1}(V_0, \infty)$. If v is larger than V_0 and the supremum of V on γ, then Σ_0, Σ_1 and γ are contained in the same connected component B of $V^{-1}(-\infty, v) \cup K$ which is a ball. Then, for the gradient vector field ∇V the two hypersurfaces Σ_0 and Σ_1 are sources while the boundary of B is the only sink. Since ∇V has no zero outside $V^{-1}(-\infty, v)$ this contradicts the Poincaré-Hopf theorem.

Note that our reasoning applies to any v larger than V_0, so the level sets $V^{-1}(v)$ are all round spheres.

From (30) we conclude that $\left|\frac{dV}{V}\right|^2$ can be expressed as a smooth function of V. We define a function $s : \mathbb{H}^n \setminus V^{-1}(V_0, \infty) \to \mathbb{R}_+$ as $s := f \circ V$ where

$$
f(v) := \int_{V_0}^{v} \frac{1}{|dV|}.
$$

Then $|ds| = 1$ so s can be interpreted as the distance function from $V^{-1}(V_0, \infty)$, see [22]. The second fundamental form of the level sets of s is given by (31) so we see that the metric g is rotationally symmetric.

Our next step is to prove that the conformal factor can be expressed as a function of s.

We remark that we can reproduce the proof of Lemma 3.2 replacing B_{R_0} by $V^{-1}(V_0 - \varepsilon, \infty)$ and find two functions $f_\pm$ solving the Yamabe equation (3) such that $f_- \leq U \leq f_+$ together with $\left|\nabla^{(k)}(f_\pm - 1)\right| \leq A_k e^{-nr}$ for any integer $k \geq 0$.

Since the conformal factor is bounded away from zero and from infinity, the metrics g and b are uniformly equivalent. Hence, taking points located further and further from $V^{-1}(V_0, \infty)$ with respect to the hyperbolic metric yields points with s going to infinity. This proves that s is unbounded.

The conformal transformation law of the mean curvature of the spheres of constant s is given by

$$H^b U^{1-\kappa} = H^g U - \frac{2(n-1)}{n-2} \partial_s U. \tag{32}$$

We choose a coordinate chart (θ^μ) on the sphere and use it to define Fermi coordinates on $\mathbb{H}^n \setminus V^{-1}(V_0, \infty)$, so that

$$g = ds^2 + f(s)\sigma_{\mu\nu} d\theta^\mu d\theta^\nu.$$

From our previous discussion, both H^g and H^b are functions of s only so we can take the derivative of Equation (32) in any μ direction,

$$(1 - \kappa)H^b U^{-\kappa} \partial_\mu U = H^g \partial_\mu U - \frac{2(n-1)}{n-2} \partial_s \partial_\mu U.$$

As s increases, the spheres of constant s become larger and larger and located further and further from $V^{-1}(V_0, \infty)$ so $H^b \to n-1$ and from Formula (32) and the estimate on U the same is true for H^g. As a consequence, the previous equation for $\partial_\mu U$ can be written as

$$\partial_s \partial_\mu U = (2 + o(1))\partial_\mu U.$$

In particular ∂_μ grows as e^{2s} unless $\partial_\mu U = 0$. Such a growth is inconsistent with the decay assumption $|\nabla U| = O(e^{-nr})$. This implies that U is constant on the level sets of s.

Without loss of generality, we can assume that the level set $V = V_0$ is a ball of radius R_1 centered at the origin of the hyperbolic space. From Propositions A.2 and A.3, there are constants m_- such that $f_{m_-}(R_1) = 0$ and m_+ such that $f_{m_+}(r) \to_{r \to R_1} \infty$. By the intermediate value theorem, there exists $m \in (m_-, m_+)$ such that $f_m(R_1)$ equals the value of U on B_{R_1}. By uniqueness of the solution of the Yamabe equation (3) with Dirichlet boundary values, we conclude that $U = f_m$ on $\mathbb{H}^n \setminus B_{R_1}$. By analytic continuation, this equality must hold everywhere on $\mathbb{H}^n \setminus K$. $\qquad\square$

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